Calculus II - Exam 1 - Fall 2013

October 3, 2013

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts or notes are not permitted. The value of each problem is indicated in brackets. Please remember the writing expectations that we've discussed in class while keeping the time constraint in mind.

1. [10 points] Each of the following functions is continuous on their respective domains. These functions can be placed in order so that the knowledge of continuity of the first in the list helps imply the continuity of the second, the second imply the third, and so on. Let a be a positive real number. In no particular order, these functions are:

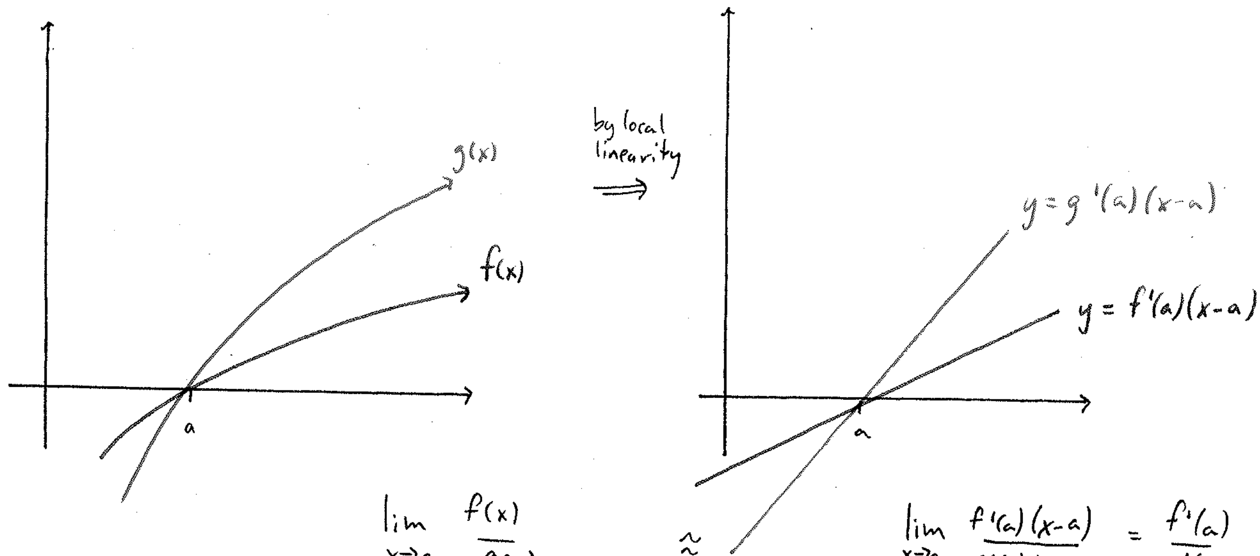
$$f(x) = \exp(x), g(x) = \log_a(x), h(x) = \frac{1}{x}, j(x) = \ln(x), k(x) = a^x.$$

Place the functions in the desired order.

We defined $\ln(x)$ as $\int_1^x \frac{1}{t}\,dt$, $x > 0$, and relied on FTC Part 1 to know that it is continuous since the integrand is (a quotient of continuous functions). Next, we defined the $\exp(x)$ as the inverse of $\ln(x)$ and knew that it was continuous by Theorem 6 of Section 6.1. Then as a^x equals $e^{x\ln a}$, we know that a^x is continuous since it is the composition of continuous functions. Finally, $\log_a x$ is the inverse of a^x and so we again apply Theorem 6 of Section 6.1. Thus, the ordering we

obtain is $\frac{1}{x}$, $\ln(x)$, $\exp(x)$, a^x, $\log_a x$.

2. [5 points] The simple form of L'Hopital's Rule states: If $f(a) = g(a) = 0$, $f'(a), g'(a)$ exist, and $g'(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$. Since a picture can be worth a thousand words, sketch a figure or two that help illustrate this statement.



$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

3. [5 points] Define what it means for a function to be **one-to-one**.

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A function f is called one-to-one if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

4. [10 points] Calculate the following limits. Identify the indeterminate form (if any).

- $\lim_{x\to 1^+} \ln(x)\tan(\pi x/2)$

The indeterminate form is $0 \cdot -\infty$. We begin by turning the product into a quotient: $\lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)}$, which now has indeterminate form $\frac{0}{0}$ and so we may apply L'Hopital's Rule.

$$\lim_{x\to 1^+} \frac{1/x}{-\csc^2(\pi x/2)\cdot \pi/2} = \lim_{x\to 1^+} \frac{2}{-\pi\csc^2(\pi x/2)}$$

$$= \frac{-2}{\pi} \lim_{x\to 1^+} \sin^2(\pi x/2) = \frac{-2}{\pi}\cdot 1 = \frac{-2}{\pi}$$

- $\lim_{x\to 0} \frac{\tan(x)-x}{x^3}$

The indeterminate form is $\frac{0}{0}$. As both numerator and denominator are differentiable, we apply L'Hopital's Rule:

$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$. Again, we have an indeterminate form of $\frac{0}{0}$, and differentiable functions.

$\lim_{x\to 0} \frac{2\sec x \cdot \sec x \cdot \tan x}{6x}$. Ditto.

$$\lim_{x\to 0} \frac{4\sec^2 x \cdot \tan^2 x + 2\sec^4 x}{6} = \frac{2}{6} = \frac{1}{3}.$$

5. [10 points] Find the second derivative with respect to x of each of the following functions:

$$G(x) = \int_3^{x^2} 2^t dt$$

By the FTC, Part I, we have $G'(x) = 2^{x^2} \cdot 2x$.

To find $G''(x)$ we use the product rule and the fact that $\frac{d(a^u)}{dx} = a^u \ln a \cdot \frac{du}{dx}$.

So,

$$G''(x) = 2^{x^2} \cdot 2 + 2x \cdot 2^{x^2} \cdot \ln 2 \cdot 2x$$

$$= 2^{x^2+1} + x^2 \cdot 2^{(x^2+2)} \cdot \ln 2$$

$$= 2^{x^2+1} \left(1 + (\ln 2) \cdot 2 \cdot x^2 \right)$$

$$y = \ln(x \cdot e^{x^2})$$

To make differentiation easier, we first take advantage of the fact that $\ln a \cdot b = \ln a + \ln b$.

So, $y = \ln x + \ln e^{x^2}$, which by facts about inverses yields

$y = \ln x + x^2$

Thus, $y' = \frac{1}{x} + 2x$ and $y'' = \frac{-1}{x^2} + 2$.

6. [10 points] Evaluate the following integrals.

- $\int 3^{\sin(x)} \cos(x) dx$

We make a u-substitution, with $u = \sin x$ and $du = \cos x \, dx$.

Thus, we obtain $\int 3^u \, du = \frac{3^u}{\ln 3} + C = \frac{3^{\sin x}}{\ln 3} + C.$

- $\int \frac{\sec(x)\tan(x)}{2+3\sec(x)} dx$

We make a u-substitution with $u = 2 + 3 \sec x \, dx$

and $du = 3 \sec x \tan x \, dx.$

Thus, we have

$$\frac{1}{3} \int \frac{3 \sec x \tan x \, dx}{2 + 3 \sec x} = \frac{1}{3} \int \frac{du}{u}$$

$$= \frac{1}{3} \ln|u| + C$$

$$= \frac{1}{3} |2 + 3\sec x| + C.$$

7. [10 points] **Carbon 14** Carbon-14, one of the three isotopes of carbon, is radioactive and decays at a rate proportional to the amount present. Its half-life is 5730 years. If 10 grams were present originally, how much will be left after 2000 years? (As calculators are not permitted, you will leave your answer in terms of powers and logs.)

As the amount of Carbon-14 decays at a rate proportional to the amount present, we may adopt the following exponential model, $y(t) = y_0 e^{kt}$, where $y(t)$ is the amount of C-14 in grams present at time t, y_0 is the initial amount of C-14, t is time in years, and k a constant.

To determine k, we know $y_0 = 10$ and that $y(5730) = 5$, since the half-life is 5730 yrs. Thus,

$$5 = 10 e^{5730k} \Rightarrow \frac{1}{2} = e^{5730k}$$

$$\Rightarrow \frac{\ln(1/2)}{5730} = k.$$

To find $y(2000)$, we simply compute:

$$y(2000) = 10 e^{\frac{\ln(1/2)}{5730} \cdot 2000} \text{ grams.}$$

8. [10 points] One of the Laws of Exponents is that $e^{x-y} = \frac{e^x}{e^y}$ for any real numbers x and y. A sketch of the proof is given below and you need to fill in some of the details.

First, $\ln(\frac{e^x}{e^y}) = \ln(e^x) - \ln(e^y)$ since

the "log of a quotient is the difference of the logs."

Now $\ln(e^x) - \ln(e^y) = x - y$ since ln and exp are inverse functions.

And, $x - y = \ln(e^{x-y})$ since , again, ln and exp are inverse functions.

Finally, we have shown that $\ln(\frac{e^x}{e^y}) = \ln(e^{x-y})$. It follows that $\frac{e^x}{e^y} = e^{x-y}$ since

ln is a one-to-one function, so we have equal inputs for equal outputs.