Calculus II - Exam 1 - Fall 2016

October 6, 2016

Name: Solution Key
Honor Code Statement: I have neither given nor received any unauthorized aid on this exam.

Total: 60 points. Avg. 43 points

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts or notes are not permitted. The value of each problem is indicated in brackets. Please remember the writing expectations that we've discussed in class while keeping the time constraint in mind.

1. [10 points] Upon retiring I will have shoved some money under my mattress. After that I will take from the money under my mattress so that the total present under the mattress decreases at the continuous rate of 10% per year. When will the mattress's stash fall to one-fifth of its initial value?

Let P denote the amount of money present under the mattress and t denote time. From the given, we know $\frac{dP}{dt} = kP$, and the solution to this differential equation is $P = P_0 e^{kt}$. We first find k given the continuous rate of decrease of 10%. At year 1 we have 90% of the initial amount.

Thus, $(.90)P_0 = P_0 e^{k(1)}$. So $.90 = e^k \Rightarrow \ln(.9) = k$.

We may now find when P equals $\frac{1}{5}P_0$.

$$\frac{1}{5}P_0 = P_0 e^{\ln(.9)t}$$

$$\frac{1}{5} = e^{\ln(.9)t}$$

$$\ln(1/5) = \ln(.9)t$$

$$\frac{\ln(.2)}{\ln(.9)} = t.$$

Thus in this # of years, I'll be at 1/5 of present value

2. [5 points each] **Find** the derivative of each of the following functions. **Indicate** the method/rule used.

(a) $p(x) = \int_{x^2}^{5} sin(2t)dt$

First $p(x) = -\int_5^{x^2} \sin(2t)\,dt$ by "reversing" interval of integration.

To find $p'(x)$, we apply the Fundamental Theorem of Calculus, Part I and the Chain Rule.

$$p'(x) = -\sin(2x^2)\cdot 2x$$

(b) $y = \log_2(x \log_5 x)$

We first convert each logarithm to the natural logarithm by the Change of Base Formula: $y = \dfrac{\ln\left(x\,\frac{\ln x}{\ln 5}\right)}{\ln 2}$. Now $\dfrac{d(\ln(u))}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$ by FTC part I and the Chain Rule. To find $\dfrac{du}{dx}$ in this case, we must employ the product rule.

Thus, $$\frac{dy}{dx} = \frac{\ln 5}{x\cdot \ln x \cdot \ln 2}\cdot \frac{d}{dx}\left(x\,\frac{\ln x}{\ln 5}\right)$$

$$= \frac{\ln 5}{\ln 2\cdot x\cdot \ln x}\cdot\frac{1}{\ln 5}\left[x\cdot\frac{1}{x} + \ln x\right]$$

$$= \frac{1}{\ln 2\cdot x\cdot \ln x}\left(1+\ln x\right).$$

$$= \frac{1}{\ln 2\cdot x\cdot \ln x} + \frac{1}{\ln 2\cdot x}$$

(c) $y = xe^{-x}$. After finding the first derivative, find the $1,000^{th}$ derivative.

To find y' we use the product rule, and established formula that $\frac{d(e^x)}{dx} = e^x$ (and so $\frac{d(e^{-x})}{dx} = -e^{-x}$ by Chain Rule).

So, $y' = -xe^{-x} + e^{-x}$

and then

$$y'' = -(-xe^{-x} + e^{-x}) + -e^{-x} = xe^{-x} - 2e^{-x}$$

$$y''' = (-xe^{-x} + e^{-x}) + 2e^{-x} = -xe^{-x} + 3e^{-x}$$

and we see a pattern established to allow us to conclude that

$$y^{1000} = xe^{-x} - 1000e^{-x}$$

3. [5 points each]

(a) State the Fundamental Theorem of Calculus, Part 2.

As given in Section 4.3 of the text:

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

(b) Define the term *one-to-one*. Give an example of a one-to-one function and justify that the example is one-to-one.

As given in Section 6.1 of the text:

A function f is called a one-to-one function if it never takes on the same value twice.

- The function $y = \ln x, x > 0$ is one-to-one since its derivative $\frac{1}{x}$ is positive for $x > 0$. This implies the function is always increasing and so cannot take on the same value twice.

A second example: $y = \frac{1}{x}$. Suppose there exists some x_1 and x_2 with the same output, i.e. $\frac{1}{x_1} = \frac{1}{x_2}$. Then, by

reciprocating both sides, we see $x_1 = x_2$.

(b)

$$\lim_{n\to\infty}(\frac{n+p}{n})^n, \quad where\ p>0$$

As p is fixed with respect to n, as $n\to\infty$, $\frac{n+p}{n}\to 1$.

Thus, we have an indeterminate form of 1^∞.

Let $y=\lim_{n\to\infty}\left(\frac{n+p}{n}\right)^n$. Then

$$\ln y = \ln \lim_{n\to\infty}\left(\frac{n+p}{n}\right)^n$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+p}{n}\right)^n$$, since the natural logarithm is a continuous function.

$$= \lim_{n\to\infty} n\ln\left(\frac{n+p}{n}\right)$$, by a property of natural log.

We have an indeterminate form of $\infty\cdot 0$. We wish to have the limit of a quotient in order to apply L'Hopital's Rule. So,

$$\ln y = \lim_{n\to\infty}\frac{\ln\left(\frac{n+p}{n}\right)}{\frac{1}{n}}$$, which has an indeterminate form of $\frac{0}{0}$, and we see that the conditions of L'Hopital's Rule are met.

$$= \lim_{n\to\infty}\frac{\ln\left(1+\frac{p}{n}\right)}{\frac{1}{n}}$$

Thus, $\ln y = \lim_{n\to\infty}\frac{\frac{n}{n+p}\cdot\frac{-p}{n^2}}{\frac{-1}{n^2}} = p\cdot\lim_{n\to\infty}\frac{n}{n+p} = p\cdot\lim_{n\to\infty}\frac{1}{1}$; by a repeated application.

Thus, $\ln y = p$, and so $y=e^p$.

4. [5 points each] Determine the following limits. Identify any indeterminate forms.

(a)

$$\lim_{s \to 0} \frac{4^s - 3^s}{s}$$

As $s \to 0$, $4^s \to 1$ and $3^s \to 1$, and so $4^s - 3^s \to 0$.

Thus we have an indeterminate form of $\frac{0}{0}$.

Note that both numerator and denominator are differentiable everywhere, and the derivative of denominator is never zero. Thus, we may apply L'Hôpital's Rule:

$$\lim_{s \to 0} \frac{4^s - 3^s}{s} = \lim_{s \to 0} \frac{4^s \ln 4 - 3^s \ln 3}{1}$$

$$= \ln 4 - \ln 3$$

5. [5 points each] Evaluate the integral.

$$\int \frac{\log_7 x}{x} dx$$

By the Change of Base Formula $\log_b a = \frac{\ln a}{\ln b}$, we may re-write the integral as $\int \frac{\ln x}{\ln 7 \cdot x} dx = \frac{1}{\ln 7} \int \frac{\ln x}{x} dx$.

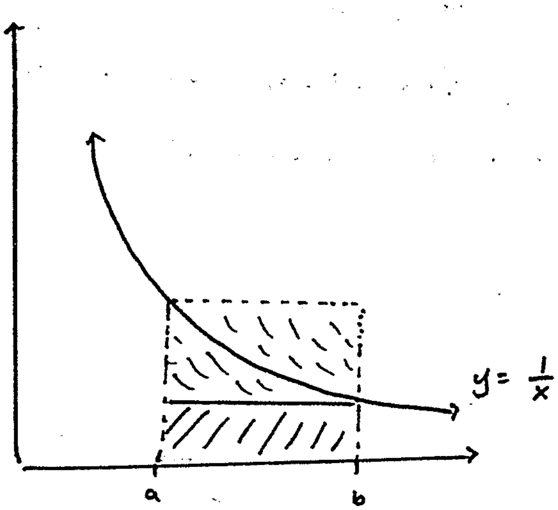
We make a u-substitution, with $u = \ln x$ and $du = \frac{1}{x} dx$. The integral has the form $\int u \, du$, with antiderivative $\frac{u^2}{2} + C$. Thus, we get.

$$\frac{1}{\ln 7} \frac{(\ln x)^2}{2} + C.$$

6. [5 points] Prove that $\lim_{x \to \infty} \ln x = \infty$.

(I strongly hinted that this question would appear. The proof is in Section 6.2*, see Equation $\boxed{4}$.)

7. [5 points] Napier's Inequality states that for $b > a > 0$, we have $\frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$. Explain how the picture below establishes this result.



The area of the smaller rectangle is $(b-a)\cdot\frac{1}{b}$.

The area under the curve is $\int_a^b \frac{1}{x}\,dx$, which equals $\ln b - \ln a$.

The area of the larger rectangle is $(b-a)\cdot\frac{1}{a}$.

The area of the smaller rectangle is less than the area under the curve, which is less than the area of the larger rectangle.

Thus, $(b-a)\frac{1}{b} < \ln b - \ln a < (b-a)\frac{1}{a}$.

Dividing through by $(b-a)$, we get the desired inequality.

Note: Many people approached this via the Mean Value Theorem for Integrals. I think that [illegible]