Calculus II - Exam 1 - Spring 2012

March 6, 2012

Name:
Honor Code Statement:

Possible: 90
Average: 69.8
S.D.: 10.1

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts or notes are not permitted. The value of each problem is indicated in brackets. Please remember the writing expectations that we've discussed in class while keeping the time constraint in mind.

1. [5 points each] Differentiate the following functions with respect to x.

- $y = \ln(\sin x)$

Applying the chain rule,

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d(\sin x)}{dx} = \frac{\cos x}{\sin x} = \cot x$$

- $y = e^{x^2+7x}$

Applying the chain rule,

$$\frac{dy}{dx} = e^{x^2+7x} \cdot \frac{d(x^2+7x)}{dx} = (2x+7)\, e^{x^2+7x}$$

- $y = \log_8(\ln x)$

First applying a change of base formula:

$$y = \frac{\ln(\ln x)}{\ln 8}$$

Next applying the chain rule,

$$\frac{dy}{dx} = \frac{1}{\ln 8} \cdot \frac{1}{\ln x} \cdot \frac{d(\ln x)}{dx} = \frac{1}{\ln 8\, x \ln x}$$

2. [8 points each] Calculate the following limits. Identify the indeterminate form (if any).

- $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$

Note: $\lim_{x\to 0} (\cos x - 1) = 0$, $\lim_{x\to 0} x^2 = 0$.

$\frac{d(\cos x - 1)}{dx} = -\sin x$ $\quad \frac{d(x^2)}{dx} = 2x$.

The indeterminate form is $0/0$, and from the above we may apply L'Hospital's Rule.

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x}$$

This limit also has indeterminate form $0/0$ and we apply L'Hospital's Rule again to obtain $\lim_{x\to 0} \frac{-\cos x}{2} = \frac{-1}{2}$.

- $\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$

Let $y = \lim_{x\to 0} (e^x + x)^{1/x}$.

We have an indeterminate form of 1^∞.

Take natural logarithm of both sides

$$\ln y = \ln \lim_{x\to 0} (e^x + x)^{1/x}$$

$= \lim_{x\to 0} \ln (e^x + x)^{1/x}$ by applying Theorem 8 of Section 1.8

$= \lim_{x\to 0} \frac{\ln(e^x + x)}{x}$

We now have an indeterminate form of $\frac{0}{0}$, with the derivative of both top and bottom existing. So, we can apply L'Hospital's Rule.

$= \lim_{x\to 0} \frac{\frac{1}{e^x + x} \cdot e^x + 1}{1}$

$= \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = 2$

And, so $\ln y = 2$, or rather $y = e^2$.

3. [5 points each] Evaluate the following integrals.

- $\int_0^{\ln 16} e^{\frac{x}{4}}\, dx$

Let $u = \frac{x}{4}$, then $du = \frac{1}{4}\, dx$. So $\int e^{x/4}\, dx = 4\int \frac{1}{4} e^{x/4}\, dx = 4e^{x/4} + C$.

Thus $\int_0^{\ln 16} e^{x/4}\, dx = 4e^{\frac{\ln 16}{4}} - 4e^0 = 4e^{\ln 16^{1/4}} - 4$

$= 4e^{\ln 2} - 4$

$= 4 \cdot 2 - 4 = 4.$

- $\int \frac{e^r}{1+e^r}\, dr$

Let $u = 1 + e^r$, then $du = e^r\, dr$.

Thus $\int \frac{e^r}{1+e^r}\, dr = \int \frac{du}{u} = \ln|u| + C$

$= \ln|1+e^r| + C.$

4. [5 points each] Define/State:

- State the Fundamental Theorem of Calculus, Part 1.

Suppose f is continuous on $[a,b]$.
If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

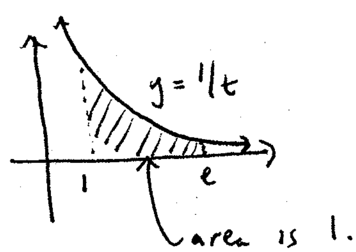
- Define the Natural Logarithm Function as an area under a curve.

$$\ln(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

- Use this definition to define the number e (Euler's constant).

e is the number such that $\ln(e) = \int_1^e \frac{1}{t}\,dt = 1$.

Here is a picture that helps explain:



5. [10 points] Use Theorem 7 of Section 6.1 of Stewart's text, the one we've talked so much about, to find the derivative of $y = \tan^{-1} x$. Justify the validity of each step.

Let $f(x) = \tan x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$. With the restriction on the domain $f(x)$ is 1-1; we already know it to be differentiable on this interval. We may apply Theorem 7 to its inverse, $f^{-1}(x) = \tan^{-1} x$.

$$(f^{-1})'(x) = \frac{1}{\sec^2(\tan^{-1}x)} \quad \text{as } f'(x) = \sec^2 x.$$

Recall the Pythagorean Identity: $\tan^2\theta + 1 = \sec^2\theta$

So, $$(f^{-1})'(x) = \frac{1}{\tan^2(\tan^{-1}x)+1} = \frac{1}{x^2+1} = \frac{d(\tan^{-1}x)}{dx}.$$

- Use your result and the chain rule to find the derivative of $y = \tan^{-1}\sqrt{2x+5}$.

By the above and the chain rule,

$$\frac{d\left(\tan^{-1}\sqrt{2x+5}\right)}{dx} = \frac{1}{(\sqrt{2x+5})^2+1} \cdot \frac{d\left(\sqrt{2x+5}\right)}{dx}$$

$$= \frac{1/2(2x+5)^{-1/2} \cdot 2}{(2x+5)+1}$$

$$= \frac{1}{\sqrt{2x+5}\cdot(2x+6)}$$

6. [10 points] **The inversion of sugar** The processing of raw sugar has a step called *inversion* that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours? (As calculators are not allowed, leave the answer in terms of powers and logs.)

Let $P(t)$ be the amount of raw sugar in kgs. at time t, where t is measured in hours.

As the rate of change is proportional to the amount present, we may write $\frac{dP}{dt} = kP$, for some constant k.

We saw that the solution of this differential equation is

$P(t) = P_0 e^{kt}$, where P_0 is the initial amount present – we are given $P_0 = 1000$ kg. So, $P(t) = 1000\, e^{kt}$.

To answer the question we must first find k. We do so knowing that $P(10) = 800$.

$$800 = 1000\, e^{10k} \iff .8 = e^{10k} \iff \frac{\ln(.8)}{10} = k.$$

We seek $P(24)$.

$$P(24) = 1000\, e^{\frac{\ln(.8)}{10} \cdot 24} \text{ kgs.}$$

$\blacksquare$

(after the exam, we see this equals $\approx$ 585 kgs.)

7. [8 points] One of the Laws of Exponents is that $e^{x+y} = e^x e^y$ for any real numbers x and y. A sketch of the proof is given below and you need to fill in some of the details.

First, $\ln(e^x e^y) = \ln(e^x) + \ln(e^y)$ since

the "log of a product equals the sum of the logs".

Now $\ln(e^x) + \ln(e^y) = x + y$ since

the natural logarithm function is the inverse of the exponential function.

And, $x + y = \ln(e^{x+y})$ since

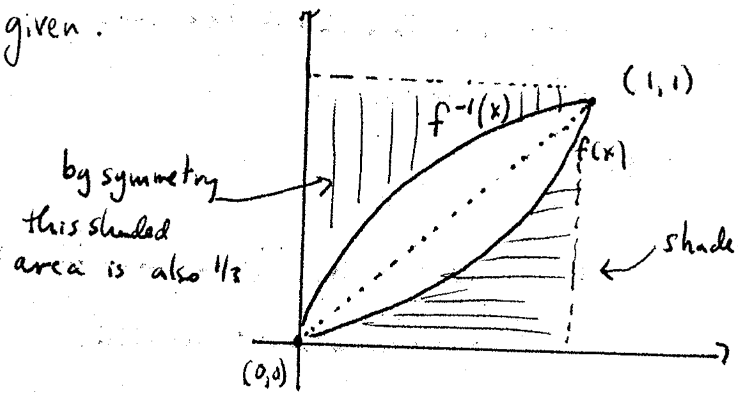
the natural logarithm function is the inverse of the exponential function.

Finally, we have shown that $\ln(e^x e^y) = \ln(e^{x+y})$. It follows that $e^x e^y = e^{x+y}$ since

the natural logarithm function is $1-1$, if the outputs are the same the inputs must also be equal.

8. [6 points] Suppose f is continuous, $f(0) = 0, f(1) = 1, f'(x) > 0$, and $\int_0^1 f(x)dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)dy$.

As $f'(x) > 0$, $f(x)$ is strictly increasing and therefore is one-to-one. As $f(x)$ is one-to-one, it has an inverse $f^{-1}(x)$. A sketch will help us determine the integral we seek. We construct it using the information given.



Note $f(x)$ lies below $y=x$

We seek $\int_0^1 f^{-1}(y)\,dy = \int_0^1 f^{-1}(x)\,dx$.

Due to symmetry, $1 - \int_0^1 f^{-1}(x)\,dx = 1/3$.

Thus $\int_0^1 f^{-1}(y)\,dy = 2/3$.

(Note: many people incorrectly decided to compute $\int_0^1 f^{-1}(x)\,dy$.)