Calculus II - Exam 1 - Spring 2013

March 7, 2013

Name:
Honor Code Statement:

65 total points

Additional Honor Code Statement: I have not observed another violating the Honor Code._____________

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. Upon completing the exam, complete and sign the honor code and the additional statement given above.

1. [10 points] **Voltage in a discharging capacitor** Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in scones, { seconds - gotta love auto-correct!

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value? (As calculators are not allowed, leave the answer in terms of powers and logs.)

We can re-write $\frac{dV}{dt} = \frac{-1}{40}V$ as $\frac{dV}{V} = \frac{-1}{40}dt$. We then integrate both sides:

$\int \frac{dV}{V} = \frac{-1}{40}\int dt$. We obtain $\ln|V| = -\frac{1}{40}t + C$. Using the exponential function,

we get $|V| = e^{-t/40 + C}$, and so $V = e^C e^{-t/40}$. When $t=0$, we have

$V_0 = e^C e^{-0/40}$, $V_0 = e^C$. Thus, $V = V_0 e^{-t/40}$.

To answer the question, we solve $.10 V_0 = V_0 e^{-t/40}$ for t.

$$.10 = e^{-t/40}$$

$$\ln(.10) = -t/40 \Rightarrow t = -40\ln(.10) \text{ seconds.}$$

2. [5 points, each]Calculate the following limits. Identify the indeterminate form (if any).

- $\lim_{x\to 0} \frac{1}{\sin(x)} - \frac{1}{x}$

indeterminate form of $\infty - \infty$

$= \lim_{x\to 0} \frac{x - \sin x}{x \sin x}$ new indeterminate form of $\frac{0}{0}$. So we can apply L'Hopital's Rule.

$= \lim_{x\to 0} \frac{1 - \cos x}{x\cos x + \sin x}$ And, again, we have indeterminate form of $\frac{0}{0}$. So we can apply L'Hopital's Rule.

$= \lim_{x\to 0} \frac{-\sin x}{\cos x + x(-\sin x) + \cos x}$

$= \frac{0}{2}$ So the limit is 0.

- $\lim_{x\to\infty} \frac{\ln(x+1)}{\log_2 x}$

We have an indeterminate form of $\frac{\infty}{\infty}$. Both numerator and denominator are differentiable, so we apply L'Hopital's Rule.

$\lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{\ln 2}\cdot\frac{1}{x}} = \lim_{x\to\infty} \ln 2 \cdot \frac{x}{x+1} = \ln 2 \lim_{x\to\infty} \frac{x}{x+1}$, and again another indeterminate form of $\frac{\infty}{\infty}$

So again apply L'Hopital's Rule.

$\ln 2 \lim_{x\to\infty} \frac{1}{1} = \ln 2.$

3. [5 points, each] Differentiate the following functions with respect to x.

- $y = 7^{x^2-x+1}$

Recall that $\dfrac{d(a^u)}{dx} = \ln a \cdot a^u \cdot \dfrac{du}{dx}$.

Thus,

$$\frac{dy}{dx} = \ln 7 \cdot 7^{x^2-x+1} \cdot (2x-1)$$

- $y = \ln(\sqrt{x})$

Recall that $\dfrac{d(\ln(u))}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}$.

Thus, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2x}$.

or note that $\ln(x^{1/2}) = \frac{1}{2} \ln x$ and then differentiate.

- $G(x) = \int_2^x \tan(t) \ln(t) dt$

Here we apply the FTC, Part 1 to obtain

$$G'(x) = \tan x \ln x.$$

4. [5 points, each] Evaluate the following integrals.

- $\int_0^4 2^x dx$ Recall $\int a^u \, du = \frac{a^u}{\ln a} + C$.

So $\int_0^4 2^x \, dx = \left. \frac{2^x}{\ln 2} \right|_0^4 = \frac{2^4}{\ln 2} - \frac{2^0}{\ln 2}$

$= \frac{1}{\ln 2} \cdot 15$

- $\int \frac{\sin(x)}{\cos(x)} dx$

This almost has the form $\int \frac{du}{u} = \ln|u| + C$.

Let $u = \cos x$, then $du = -\sin x \, dx$

Thus

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + C$$

5. Define/State:

See page 209.

- [5 points] State the Mean Value Theorem and draw a picture that helps illustrate the statement.

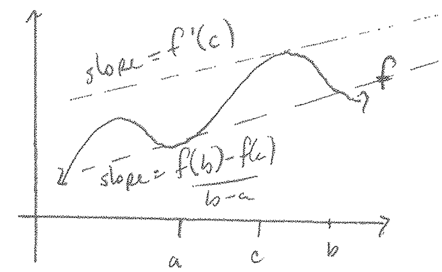
Let f be a function that satisfies the following hypotheses:

① f is continuous on $[a,b]$

② f is differentiable on (a,b).

Then there is a $c \in (a,b)$ such that

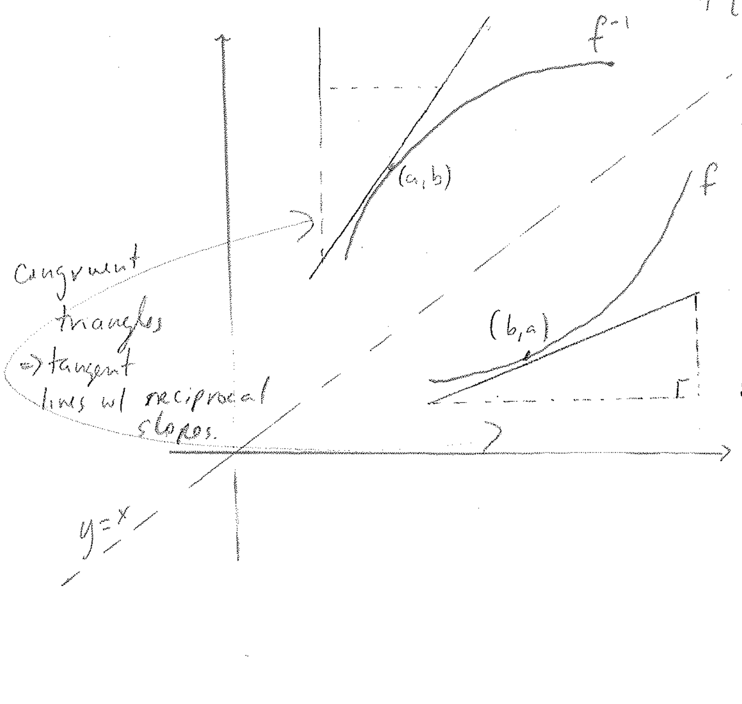
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



- [5 points] Give the statement of Theorem 7 of Section 6.1 and draw a picture that helps illustrate the statement.

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$



6. [10 points] One of the Laws of Logarithms is: if x is a positive real number and r a rational number, then $\ln(x^r) = r\ln(x)$. Help finish the proof, a sketch of which is below, by filling in the blanks.

PROOF: Let $f(x) = \ln(x^r)$ and $g(x) = r\ln(x)$.

We have $f'(x) =$ $\frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x}$

We also have $g'(x) =$ $r \cdot \frac{1}{x}$

Since these derivatives are the same, $f(x)$ and $g(x)$ differ by a constant.
That is,

$$\ln(x^r) + C = r\ln(x).$$

Now, let $x = 1$. We get $\ln(1^r)+C = r\ln(1)$. So by the fact that $\ln(1) = 0$
C equals 0.

Thus, $\ln(x^r) = r\ln(x)$.