Calculus II - Exam 2 - Techniques of Integration

October 17, 2013

75 points total

Avg. score $\frac{62.8}{75}$

Name: Answer Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Additional Honor Statement: I have not observed another violating the Honor Code.________

Directions: Justify all answers/solutions. Calculators are not permitted. You may use the table of trigonometric identities given on the last page. Each problem is worth 10 points, except for the last, which is worth 5 points. If you need extra space, use the blank white paper provided.

1.

$$\int \frac{4x}{36-x^2}dx$$

We make a u-substitution: $u = 36 - x^2$, $du = -2x$.

Thus the integral becomes

$$-2\int \frac{-2x}{36-x^2}\,dx = -2\int \frac{du}{u} = -2\ln|u| + C$$

$$= -2\ln|36-x^2| + C.$$

One might also approach this problem via partial fraction decomposition or trigonometric substitution.

2.

$$\int \frac{\cos(x) + \sin(2x)}{\sin(x)} dx$$

Mixed arguments cause headaches, so we take advantage of a double-angle formula: $\sin(2x) = 2 \sin x \cos x$.

So the integral equals:

$$\int \frac{\cos x + 2 \sin x \cos x}{\sin x} dx$$

$$= \int \frac{\cos x}{\sin x} dx + 2 \int \cos x \, dx$$

$$= \ln|\sin x| + 2 \sin x + C.$$

3.

$$\int_0^{2\pi} t^2 \sin(2t)dt$$

We use integration by parts:

$$u = t^2 \qquad dv = \sin 2t \, dt$$

$$du = 2t\, dt \qquad v = -\frac{1}{2}\cos 2t$$

Thus,

$$\int_0^{2\pi} t^2 \sin(2t)\, dt = -\frac{t^2}{2}\cos(2t)\Big|_0^{2\pi} - \int_0^{2\pi} -t\cos(2t)\, dt$$

$$= -\frac{t^2}{2}\cos(2t)\Big|_0^{2\pi} + \int_0^{2\pi} t\cos(2t)\, dt$$

Again we need to use integration by parts:

$$u = t \qquad dv = \cos(2t)\, dt$$

$$du = 1\, dt \qquad v = \frac{1}{2}\sin(2t)$$

And so we obtain

$$-\frac{t^2}{2}\cos(2t)\Big|_0^{2\pi} + \frac{t}{2}\sin(2t)\Big|_0^{2\pi} - \int_0^{2\pi}\frac{1}{2}\sin(2t)\, dt$$

$$= -\frac{t^2}{2}\cos(2t)\Big|_0^{2\pi} + \frac{t}{2}\sin(2t)\Big|_0^{2\pi} + \frac{1}{4}\cos(2t)\Big|_0^{2\pi}$$

$$= \left(-\frac{4\pi^2}{2} - 0\right) + (0 - 0) + \left(+\frac{1}{4} - +\frac{1}{4}\right)$$

$$= -2\pi^2$$

4. What is the partial fraction decomposition of the following quotient? (Note that I'm not asking for the antiderivative.)

$$\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}$$

Note that x^2+1 is an irreducible quadratic.

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2. \quad (*)$$

Let $x=1$, then

$$-2 = B(2) \Rightarrow B = -1.$$

Let $x=0$, then

$-1 = -A + B + D$ and as $B=-1$, we have $A = D$.

We expand equation $(*)$ and equate coefficients.

$$x^2-2x-1 = Ax^3 - Ax^2 + Ax - A + Bx^2 + B + (Cx+D)(x^2-2x+1)$$

$$x^2-2x-1 = Ax^3 - Ax^2 + Ax - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D$$

The coefficient of x^3 on the left is 0 and on the right is $A+C$

.. .. .. x^2 .. .. .. 1 and .. .. .. $-A+B-2C+D$

.. .. .. x .. .. .. -2 .. .. $A+C-2D$

.. .. constant .. .. -1 .. .. $-A+B+D$

So we have a system of equations

$$\begin{aligned}
0 &= A + C \\
1 &= -A + B - 2C + D \\
-2 &= A + C - 2D \\
-1 &= -A + B + D
\end{aligned}$$

Knowing that $B = -1$ and $A = D$, I can simplify this system to

$$\begin{aligned}
0 &= A + C \\
1 &= -A - 1 - 2C + A \quad \Rightarrow \quad 2 = -2C \quad \Rightarrow C = -1 \\
-2 &= A + C - 2A \\
-1 &= -A - 1 + A
\end{aligned}$$

The second equation yields a value of C as noted.
I plug C into the first: $0 = A - 1 \Rightarrow A = 1$, and since $A = D$, we have $D = 1$.

Thus the decomposition is

$$\frac{1}{(x-1)} + \frac{-1}{(x-1)^2} + \frac{-1x+1}{(x^2+1)}.$$

<u>Note</u>: There might be a shorter/quicker approach than what I've given here.

5. Determine if the following integral converges. If it converges, determine to what it converges.

$$\int_{-2}^{14} \frac{dx}{\sqrt{x+2}}$$

Since $\frac{1}{\sqrt{x+2}}$ has a discontinuity at $x=-2$, this is a Type II improper integral. The integral equals

$$\lim_{b \to -2^+} \int_b^{14} \frac{dx}{\sqrt{x+2}}.$$

To find an antiderivative, let $u = x+2$, $du = 1\,dx$

And note that $\int \frac{dx}{\sqrt{x+2}} = \int u^{-1/2}\,du = 2u^{1/2} + C$.

Thus this limit equals

$$\lim_{b \to -2^+} 2\sqrt{x+2}\Big|_b^{14} = \lim_{b \to -2^+} \left(2\sqrt{16} - 2\sqrt{b+2}\right)$$

$$= 2 \cdot 4 - 2 \cdot 0$$

$$= 8$$

The integral converges to 8.

6. Determine if the following integral converges. If it converges, determine to what it converges.

$$\int_0^\infty \frac{x}{x^2+9}dx$$

This is a Type I improper integral; it equals

$$\lim_{b\to\infty} \int_0^b \frac{x}{x^2+9}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b \frac{2x}{x^2+9}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln\left|x^2+9\right|\Big|_0^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln(b^2+9) - \frac{1}{2}\ln(9)$$

$$= \infty.$$

That is, the integral diverges.

7. Use the Direct Comparison Test to determine whether or not the following integral converges.

$$\int_{103}^{\infty} \frac{1}{x^2 + 5x - 109} dx$$

We compare this to the convergent p-integral, $\int_{103}^{\infty} \frac{1}{x^2} dx$.

Note that since $\int_{1}^{\infty} \frac{1}{x^2} dx$ converges, so does $\int_{103}^{\infty} \frac{1}{x^2} dx$.

Now we show: $\dfrac{1}{x^2+5x-109} \leq \dfrac{1}{x^2}$

This holds iff $x^2 \leq x^2 + 5x - 109$

$\iff 0 \leq 5x - 109$

$\iff \frac{109}{5} \leq x$. This holds!

Thus by a direct comparison, the given integral converges.

8. When we considered p-integrals, we restricted our attention to $p > 0$. Why did we restrict our attention in this manner?

Recall that p-integrals are of the form $\int_1^\infty \frac{1}{x^p}\,dx$. Now, if $p \le 0$, then the question of convergence is rather trivial, as explained below.

First consider if $p=0$. The integral is $\int_1^\infty \frac{1}{x^0}\,dx = \int_1^\infty 1\,dx$

$= \lim_{b\to\infty} \int_1^b 1\,dx = \lim_{b\to\infty} x\Big|_1^b$, which is a divergent limit.

Now if $p<0$, then we may rewrite the integral as $\int_1^\infty x^m\,dx$, where $m>0$. Now $\int_1^\infty x^m\,dx = \lim_{b\to\infty}\int_1^b x^m\,dx$

$= \lim_{b\to\infty} \frac{x^{m+1}}{m+1}\Big|_1^b$, which is also a divergent limit.

So we restricted our attention to $p>0$ in order to have a question worth the challenge.

Some common errors on the exam:

(1) Not using notation when it is called for.

For example, someone may have written something like

$$\int \frac{\cos x}{\sin x}\, dx + 2\int \cos x\, dx$$

$$\ln|\sin x| + 2\sin x + C.$$

Here the person should have used equal signs to "connect" what's written on the page.

Another example is this:

$$\lim_{b\to -2^+} \int_b^{14} \frac{dx}{\sqrt{x+2}} = 2\sqrt{x+2}\,\Big|_b^{14}$$

Here the person "dropped" the limit sign.

(2) When discussing integrals, many people wrote something like "$\frac{1}{x^p}$ converges when $p > 1$". It's not the integrand that converges, but it is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ that does.

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$