Calculus II - Exam 2 - Techniques of Integration

October 20, 2016

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Additional Honor Statement: I have not observed another violating the Honor Code.______________

Total points 70

Average 48.25 pts.

SD 10.7 pts.

Directions: Justify all answers/solutions. For each of the first five problems, find the integral and **identify** the technique of integration used. Calculators are not permitted. You may use the table of trigonometric identities given on the last page. Each problem is worth 10 points. If you need extra space, use the blank white paper provided.

1.

$$\int \sin(2\theta)\sin(6\theta)d\theta$$

We use the penultimate identity given in the last page to rewrite the integral as (note that $\sin 2\theta \sin 6\theta = \sin 6\theta \sin 2\theta$, so we may let $A = 6\theta$, $B = 2\theta$)

$$\int \frac{1}{2}\cos 4\theta - \frac{1}{2}\cos 8\theta \, d\theta$$

$$= \frac{1}{2}\frac{1}{4}\sin 4\theta - \frac{1}{2}\cdot\frac{1}{8}\sin 8\theta + C$$

$$= \frac{1}{8}\sin 4\theta - \frac{1}{16}\sin 8\theta + C.$$

2.

$$\int \frac{(\ln x)^2}{x^3}dx$$

We'll try using integration by parts.

Let $u = (\ln x)^2$ and $dv = \frac{dx}{x^3}$. Then $du = 2\ln x \cdot \frac{1}{x}dx$ and $v = \frac{1}{-2x^2}$. So the integral becomes,

$$(\ln x)^2 \cdot \frac{1}{-2x^2} - \int \frac{2\ln x}{-2x^3}dx = \frac{(\ln x)^2}{-2x^2} + \int \frac{\ln x}{x^3}dx =$$

And we use integration by parts, again.

Let $u = \ln x$ and $dv = \frac{dx}{x^3}$. Then $du = \frac{1}{x}dx$ and $v = \frac{1}{-2x^2}$.

We obtain

$$-\frac{(\ln x)^2}{2x^2} + \left[\frac{\ln x}{-2x^2} - \int \frac{1}{-2x^3}dx\right]$$

$$= -\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} + \frac{1}{2}\left(\frac{1}{-2x^2}\right) + C$$

$$= -\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

3.

$$\int_0^1 \sqrt{x-x^2}dx$$

(Hint: begin by completing the square.)

We follow the hint.

$$x - x^2 = -(x^2 - x) = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4}$$

$$= -\left(x - 1/2\right)^2 + 1/4$$

So the integral becomes:

$\int_0^1 \sqrt{1/4 - (x-1/2)^2}\, dx$. We now do a trig substitution.

Let $x - \frac{1}{2} = \frac{\sin\theta}{2}$. Then $dx = \frac{1}{2}\cos\theta\, d\theta$. Also $2x - 1 = \sin\theta$, so if $x=0$, then $\sin\theta = -1$, i.e $\theta = -\frac{\pi}{2}$; and if $x = 1$, then $\sin\theta = 1$, i.e $\theta = \frac{\pi}{2}$.

So the integral becomes,

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \sqrt{\frac{1}{4} - \frac{\sin^2\theta}{4}} \cdot \frac{1}{2}\cos\theta\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{\frac{1}{4}(1-\sin^2\theta)}\, \frac{1}{2}\cos\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$, by half-angle identity.

$$= \frac{1}{4}\left[\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right]\Bigg|_{-\pi/}^{\pi/2}$$

4.

$$\int \frac{x}{(x+4)(2x-1)}dx$$

Let's do a partial fraction decomposition.

$$\frac{x}{(x+4)(2x-1)} = \frac{A}{(x+4)} + \frac{B}{(2x-1)}$$

$$x = A(2x-1) + B(x+4)$$

Let $x=-4$, then $-4 = A(-9) \Rightarrow A = \frac{4}{9}$

Let $x=\frac{1}{2}$, then $1/2 = B(9/2) \Rightarrow B = 1/9$.

So, the integral becomes

$$\int \frac{4/9}{(x+4)}dx + \int \frac{1/9}{(2x-1)}dx$$

$$= \frac{4}{9}\int \frac{dx}{x+4} + \frac{1}{9}\int \frac{dx}{2(x-1/2)}$$

$$= \frac{4}{9}\ln|x+4| + \frac{1}{18}\ln|x-1/2| + C.$$

5.

$$\int_0^\infty \sin x \cdot e^{\cos x} dx$$

By definition, this improper integral equals

$$\lim_{b\to\infty} \int_0^b \sin x \cdot e^{\cos x} dx$$

To evaluate the integral, we do a u-substitution with $u = \cos x$ and $du = -\sin x\, dx$. If $x=0$, then $u=\cos(0)=1$

If $x=b$, then $u=\cos(b)$. Thus, we have

$$\lim_{b\to\infty} -\int_1^{\cos b} e^u\, du$$

$$= \lim_{b\to\infty} -e^u \Big|_1^{\cos b} = \lim_{b\to\infty} -e^{\cos b} + e^1$$

As $b \to \infty$, $\cos(b)$ oscillates between 1 and -1.

Thus this limit does not exist, i.e. the integral is divergent

6. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$$

Note that since $-1 \le \sin x \le 1$ for all x, we have $0 \le \sin^2 x \le 1$ for all x. Thus, $\frac{\sin^2 x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for $x > 0$.

Thus, $0 \le \int_0^{\pi} \frac{\sin^2 x \, dx}{\sqrt{x}} \le \int_0^{\pi} \frac{1}{\sqrt{x}} dx$. So we will

Consider the integral on the right side of the above inequality.

$\int_0^{\pi} \frac{1}{\sqrt{x}} dx$ is an improper integral, which may be expressed

as $\lim\limits_{b \to 0^+} \int_b^{\pi} x^{-1/2} dx = \lim\limits_{b \to 0^+} 2x^{1/2} \Big|_b^{\pi} = 2\sqrt{\pi} - \lim\limits_{b \to 0^+} 2\sqrt{b}$

$= 2\sqrt{\pi}$

Thus, since this integral converges, the given one does as well as the result of the Comparison Theorem.

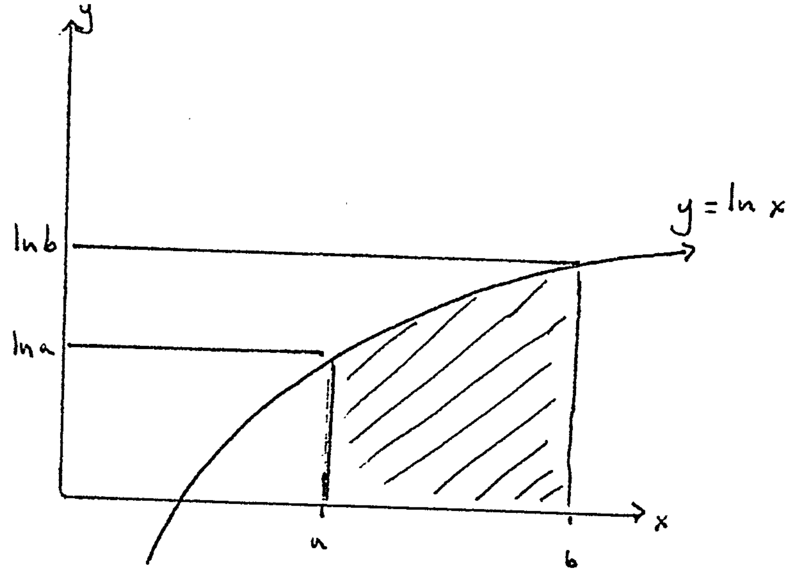
$$= \frac{1}{8}\theta + \frac{1}{16}\sin 2\theta \Big|_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{\pi}{16} + \frac{1}{16}\sin\pi\right) - \left(\frac{-\pi}{16} + \frac{1}{16}\sin(-\pi)\right)$$

$$= \frac{\pi}{8}.$$

7. Explain the following figure and the equation that goes with it.

$$\int_a^b \ln x\, dx \overset{(1)}{=} b\ln b - a\ln a - \int_{\ln a}^{\ln b} e^y\, dy \overset{(2)}{=} (x\ln x)|_a^b - (b-a) \overset{(3)}{=} (x\ln x - x)|_a^b$$



We seek the area under the curve $y = \ln x$ between a and b – this corresponds to the expression $\int_a^b \ln x\, dx$ and the shaded area in the figure. One may find or express this area in the following manner: it is the area of the larger rectangle minus the two following areas, the area of the small rectangle and the area "to the left" of $y = \ln x$. The area of the large rectangle is $b \ln b$, the area of the small rectangle is $a \ln a$ and the area "to the left" of the curve can be found by integrating with respect to y, i.e. $\int_{\ln a}^{\ln b} e^y\, dy$. Thus, we have equality (1).

Equality (2) may be obtained by re-expressing the 1st two terms in short-hand and the integral obtained (quite easily) and evaluated. Finally, (3) is obtained via simply a shorthand ~~exp~~ re-expression.

The figure thus illustrates that the indefinite integral of $\ln x$ is

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$