Calculus II - Exam 2 - Spring 2012
Techniques of Integration

March 22, 2012

Name:
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

70 points total.
Directions: Justify all answers/solutions. Calculators are not permitted. You may use the table of trigonometric identities given on the last page. Each problem is worth 10 points. If you need extra space, use the blank white paper provided. You may omit one of Question 4 or 5.

1. Evaluate each of the following integrals. If there is a particular technique that you use, name it.

   (a) $\int x \sec^2(x)\, dx$

We use integration by parts: let $u = x$, $dv = \sec^2 x\, dx$. Then $du = 1dx$, $v = \tan x$.

Thus, $\int x \sec^2 x\, dx = x \cdot \tan x - \int \tan x\, dx$.

Now, $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$ and by a u-substitution $u = \cos x$, $du = -\sin x\, dx$ this antiderivative is $-\ln|\cos x| + C$.

Thus $\int x \sec^2 x\, dx = x\tan x + \ln|\cos x| + C$.

   (b) $\int \tan^2 x \sec^4 x\, dx$

$$= \int \tan^2 x \cdot \sec^2 x \cdot \sec^2 x\, dx$$

$$= \int \tan^2 x\,(\tan^2 x + 1) \cdot \sec^2 x\, dx,$$ by a Pythagorean Identity.

$$= \int (\tan^4 x + \tan^2 x) \cdot \sec^2 x\, dx,$$ We now use a u-substitution where $u = \tan x$, $du = \sec^2 x\, dx$

$$= \int u^4 + u^2\, du$$

$$= \frac{u^5}{5} + \frac{u^3}{3} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.$$

(c) $\int_0^1 \frac{1}{(x^2+1)^2}\, dx$ (Hint: use a trigonometric substitution.)

We follow the hint

Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = \sec^2\theta\, d\theta$.

If $x = 0$, then $0 = \tan\theta$ and $\theta = 0$.

If $x = 1$, then $1 = \tan\theta$ and $\theta = \frac{\pi}{4}$.

Via this substitution, we obtain

$$\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta + 1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2} = \int_0^{\pi/4} \cos^2\theta\, d\theta,$$

which via a half-angle formula becomes

$$\int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \Big|_0^{\pi/4}$$

$$= \left(\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(\frac{1}{2}\cdot 0 + \frac{1}{4}\sin 0\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}.$$

(d) $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

First we perform polynomial long-division on the integrand.

$$\begin{array}{r} x+1 \\ x^2-x-6\ \big)\ \overline{x^3+0x^2-4x-10} \\ -(x^3-x^2-6x) \\ \hline x^2+2x-10 \\ -(x^2-x-6) \\ \hline 3x-4 \end{array}$$

Thus, the integral becomes:

$$\int_0^1 x+1+\frac{3x-4}{(x^2-x-6)}\,dx = \int_0^1 x+1+\frac{3x-4}{(x-3)(x+2)}\,dx.$$

We now perform a partial fraction decomposition.

$$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2} \Rightarrow 3x-4 = Ax+2A+Bx-3B$$

so
$$A+B=3$$
$$2A-3B=-4$$

As $A=3-B$, we substitute $2(3-B)-3B=-4$
$$6-5B=-4$$
$$-5B=-10 \qquad B=2 \text{ and so } A=1.$$

The integral is thus.

$$\int_0^1 x+1+\frac{1}{x-3}+\frac{2}{x+2}\,dx = \frac{x^2}{2}+x+\ln|x-3|+2\ln|x+2|\Big|_0^1$$
$$=\left(\frac{1}{2}+1+\ln 2+2\ln 3\right)-\left(\ln 3+2\ln 2\right)$$
$$=\frac{3}{2}-\ln 2+\ln 3$$

2. Determine whether the following integrals converge or diverge.

$$\int_e^\infty \frac{1}{x(\ln(x))^3}\,dx$$

By definition, this equals $\lim\limits_{b\to\infty} \int_e^b \frac{1}{x(\ln x)^3}\,dx$. To find the indefinite integral we use integration by u-substitution.

Let $u = \ln x$, $du = \frac{1}{x}$. Note if $x = e$ then $u = \ln e = 1$ and if $x = b$ then $u = \ln b$

The integral becomes.

$$\lim_{b\to\infty} \int_1^{\ln b} \frac{1}{u^3}\,du = \lim_{b\to\infty} \left.\frac{-1}{2u^2}\right|_1^{\ln b} = \lim_{b\to\infty} \frac{-1}{2(\ln b)^2} - \frac{-1}{2\cdot 1} = \frac{1}{2}.$$

The integral converges; it converges to $\frac{1}{2}$.

3. Use the Direct Comparison Theorem to determine whether the following integral is convergent or divergent.

$$\int_1^\infty \frac{2+\sin(x)}{x^2}\,dx$$

We will compare to the convergent p-integral $\int_1^\infty \frac{1}{x^2}\,dx$, and the convergent integral $\int_1^\infty \frac{3}{x^2}\,dx$.

Note that as $-1 \le \sin x \le 1$ for all real x, $1 \le 2+\sin x \le 3$.

Thus,

$$\int_1^\infty \frac{1}{x^2}\,dx \le \int_1^\infty \frac{2+\sin x}{x^2}\,dx \le \int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx.$$

So as $\int_1^\infty \frac{1}{x^2}\,dx$ converges, the given integral also converges.

4. Find the area of the region bounded by the given curves.

$$y = x^2 e^{-x}, \quad y = xe^{-x}$$

First we find where these curves intersect.

$$x^2 e^{-x} = xe^{-x} \iff x^2 = x \iff x^2 - x = 0 \iff x(x-1) = 0.$$

So, they intersect at $x=0$ and $x=1$.

On the interval $[0,1]$ we have $x^2 \leq x$, thus $x^2 e^{-x} \leq xe^{-x}$.

Also, note $0 \leq x^2 e^{-x}$.

So we seek $\displaystyle\int_0^1 xe^{-x} - x^2 e^{-x}\, dx$.

We use integration by parts:

Let $u_1 = x$ $\quad du_1 = 1\, dx$ $\qquad u_2 = x^2$ $\quad du_2 = 2x\, dx$

$dv_1 = e^{-x} dx$ $\quad v = -e^{-x}$ $\qquad dv_2 = e^{-x} dx$ $\quad v = -e^{-x}$

Thus,

$$\int_0^1 xe^{-x} - x^2 e^{-x} = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx - \left(-x^2 e^{-x}\Big|_0^1 + \int_0^1 2x e^{-x}\,dx\right)$$

$$= -xe^{-x} - e^{-x} + x^2 e^{-x}\Big|_0^1 - 2\int_0^1 x e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + x^2 e^{-x} - 2\left(-xe^{-x} - e^{-x}\right)\Big|_0^1$$

$$= xe^{-x} + e^{-x} + x^2 e^{-x}\Big|_0^1$$

$$= \left(\frac{1}{e} + \frac{1}{e} + \frac{1}{e}\right) - (0 + 1 + 0) = \frac{3}{e} - 1.$$

5. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of ~~x~~ f on the interval $[a, \infty)$ to be

$$\lim_{t \to \infty} \frac{1}{t-a} \int_a^t f(x)\, dx.$$

Find the average value of $f(x) = \sin(x)$ on the interval $[0, \infty)$.

By the above definition the average value equals

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sin x \, dx$$

$$= \lim_{t \to \infty} \frac{1}{t} \left(-\cos x\right)\Big|_0^t$$

$$= \lim_{t \to \infty} \frac{-1}{t} \cos t - \frac{-\cos(0)}{t}$$

$$= \lim_{t \to \infty} \frac{-\cos t + 1}{t}$$

$= 0$ by the Squeeze Theorem, since $\frac{0}{t} \leq \frac{-\cos t + 1}{t} \leq \frac{2}{t}$

and $\lim_{t \to \infty} \frac{0}{t} = \lim_{t \to \infty} \frac{2}{t} = 0.$

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$