Calculus II - Exam 2 - Techniques of Integration

March 21, 2013

Total points: 80

Name: Key

Honor Code Statement: I have neither given nor received any unauthorized aid on this exam.

Additional Honor Statement: I have not observed another violating the Honor Code.______

Directions: Justify all answers/solutions. Calculators are not permitted. You may use the table of trigonometric identities given on the last page. Each problem is worth 10 points. If you need extra space, use the blank white paper provided.

1. $\int_0^{2\pi} \sqrt{\frac{1-\cos 2x}{2}}\, dx$

We begin by taking advantage of a half-angle formula, $\frac{1-\cos(2x)}{2} = \sin^2 x$.

So the integral becomes,

$$\int_0^{2\pi} \sqrt{\sin^2 x}\, dx = \int_0^{2\pi} \sin(x)\, dx = -\cos x \Big|_0^{2\pi}$$

$$= -\cos(2\pi) + \cos(0)$$

$$= -1 + 1 = 0$$

2. $\int_1^e x^3 \ln x dx$

We use integration by parts, $\int u\,dv = uv - \int v\,du$.

Let $u = \ln x$, $dv = x^3\,dx$.

Then $du = \frac{1}{x}\,dx$ $\quad v = \frac{x^4}{4}$

Thus,

$$\int_1^e x^3 \ln x \, dx = \frac{x^4}{4} \cdot \ln x \Big|_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^4}{4} \cdot \ln x \Big|_1^e - \frac{1}{4}\int_1^e x^3\,dx$$

$$= \frac{x^4}{4} \cdot \ln x \Big|_1^e - \frac{1}{16} x^4 \Big|_1^e$$

$$= \left(\frac{e^4}{4} \cdot \ln e - \frac{1}{16} e^4\right) - \left(\frac{1}{4}\ln 1 - \frac{1}{16}\right)$$

$$= \frac{4e^4 - e^4}{16} + \frac{1}{16}$$

$$= \frac{3e^4 + 1}{16}$$

3. $\int e^{-2x}\sin(2x)dx$

We use integration by parts, $\int u\,dv = uv - \int v\,du$

Let $u = e^{-2x}$ $\quad dv = \sin(2x)\,dx$

$du = -2e^{-2x}dx \quad v = -\frac{1}{2}\cos(2x)$

Then

$$\int e^{-2x}\sin(2x)\,dx = -\frac{1}{2}e^{-2x}\cos(2x) - \int e^{-2x}\cos(2x)\,dx$$

For this new integral, we use integration by parts again.

Let $u = e^{-2x}$ $\quad dv = \cos(2x)\,dx$

$du = -2e^{-2x}dx \quad v = \frac{1}{2}\sin(2x)$

Thus,

$$\int e^{-2x}\sin(2x)\,dx = -\frac{1}{2}e^{-2x}\cos(2x) - \left[e^{-2x}\cdot\frac{1}{2}\sin(2x) - \int -e^{-2x}\sin(2x)\,dx\right]$$

$$= -\frac{1}{2}e^{-2x}\left(\cos(2x) + \sin(2x)\right) - \int e^{-2x}\sin(2x)$$

So,

$$2\int e^{-2x}\sin(2x)\,dx = -\frac{1}{2}e^{-2x}\left(\cos 2x + \sin 2x\right) + C$$

$$\int e^{-2x}\sin(2x)\,dx = -\frac{1}{4}e^{-2x}\left(\cos 2x + \sin 2x\right) + C.$$

4. $\int \frac{2x+1}{x^2-7x+12}dx$

We will perform a partial fraction decomposition.

$$\frac{2x+1}{x^2-7x+12} = \frac{2x+1}{(x-3)(x-4)} = \frac{A}{(x-3)} + \frac{B}{(x-4)}$$

Thus, $2x+1 = A(x-4) + B(x-3)$.

Let $x=4$ to solve for B.

$$2\cdot 4+1 = A\cdot 0 + B\cdot 1$$

$$9 = B$$

Let $x=3$ to solve for A.

$$2\cdot 3+1 = A(-1) + B\cdot 0$$

$$7 = -A$$

$$-7 = A$$

Thus,

$$\int \frac{2x+1}{x^2-7x+12}dx = \int \frac{-7}{x-3}dx + \int \frac{9}{x-4}dx$$

$$= -7\ln|x-3| + 9\ln|x-4| + C$$

5. $\int \frac{x^4}{x^2-1}dx$

We first perform polynomial long division

$$\begin{array}{r l} & x^2+1 \quad R\,1 \\ x^2-1 \, \big) & x^4 \\ & -(x^4-x^2) \\ \hline & x^2 \\ & -(x^2-1) \\ \hline & +1 \end{array}$$

Thus $\dfrac{x^4}{x^2-1} = x^2+1 + \dfrac{1}{x^2-1}$

So,

$$\int \frac{x^4}{x^2-1}\,dx = \int x^2+1+\frac{1}{x^2-1}\,dx$$

$$= \int x^2+1+\frac{1}{(x+1)(x-1)}\,dx$$

$$= \frac{x^3}{3} + x + \underbrace{\int \frac{1}{(x+1)(x-1)}\,dx}$$

We can do a partial fraction decomposition (or take advantage of a memorized formula)

$$\frac{1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$$1 = A(x-1) + B(x+1)$$

So, $B = \frac{1}{2}$, $A = \frac{-1}{2}$

$$\int \frac{x^4}{x^2-1}\,dx = \frac{x^3}{3} + x + \int \frac{-\frac{1}{2}}{(x+1)}\,dx + \int \frac{\frac{1}{2}}{(x-1)}\,dx$$

$$= \frac{x^3}{3} + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$= \frac{x^3}{3} + x + \frac{1}{2}\ln\frac{|x-1|}{|x+1|} + C$$

6. $\int 35\sin^4 x\cos^3 x dx$

We have an even number of sin(x), and an odd number of cos(x). We'll "reserve" one cos(x)dx for the du and let $u = \sin x$.

$$= \int 35 \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$$

by a Pythagorean Identity.

$$= 35 \int \sin^4 x \cdot (1-\sin^2 x) \cdot \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= 35 \int u^4 (1-u^2) \, du$$

$$= 35 \int u^4 - u^6 \, du$$

$$= 35 \left(\frac{u^5}{5} - \frac{u^7}{7} \right) + C$$

$$= 35 \left(\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} \right) + C$$

$$= 7\sin^5 x - 5\sin^7 x + C$$

7. $\int_0^{1/3} \sqrt{1-9t^2}\,dt$

We do a trig. substitution in hopes of eliminating the square root.

Let $t = \frac{1}{3}\sin\theta$, so $dt = \frac{1}{3}\cos\theta \, d\theta$

We restrict θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

If $t=0$, then $0 = \frac{1}{3}\sin\theta \Rightarrow \sin^{-1}(0) = \theta \Rightarrow \theta = 0$.

If $t = 1/3$, then $\frac{1}{3} = \frac{1}{3}\sin\theta \Rightarrow \sin^{-1}(1) = \theta \Rightarrow \theta = \frac{\pi}{2}$.

Thus,

$$\int_0^{1/3} \sqrt{1-9t^2}\,dt = \int_0^{\pi/2} \sqrt{1-9\left(\frac{1}{3}\sin\theta\right)^2}\;\frac{1}{3}\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta$$

by a Pythagorean Identity

$$= \frac{1}{3}\int_0^{\pi/2} \sqrt{\cos^2\theta}\;\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{3}\cdot\left(\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{6}\theta + \frac{1}{12}\sin 2\theta$$

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$$= \frac{1}{6}\cdot\frac{\pi}{2} + \frac{1}{12}\sin(\pi) - \left(0 + \frac{1}{12}\sin 0\right)$$

$$= \frac{\pi}{12}$$

Determine if the following integral converges.

8. $\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx$ This is a Type I improper integral.

We give two different approaches to this problem. In the first, we find an antiderivative. In the second, we make a comparison.

① $$\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx = \lim_{b\to-\infty} \int_b^2 \frac{2}{x^2+4}\,dx = 2\lim_{b\to-\infty} \int_b^2 \frac{1\,dx}{4\left(\left(\frac{x}{2}\right)^2+1\right)} = 2\lim_{b\to-\infty} \frac{1}{4}\int_b^2 \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

Now let $u = \frac{x}{2}$, then $du = \frac{1}{2}dx$, so the integral becomes

$$\frac{1}{2}\lim_{b\to-\infty} 2\int_{\frac{b}{2}}^{1} \frac{du}{u^2+1} = \lim_{b\to-\infty} \int_{\frac{b}{2}}^{1} \frac{du}{u^2+1} = \lim_{b\to-\infty} \tan^{-1}(u)\Big|_{\frac{b}{2}}^{1}$$

$$= \lim_{b\to-\infty} \tan^{-1}(1) - \tan^{-1}\left(\frac{b}{2}\right) = \frac{\pi}{4} - \left(\frac{-\pi}{2}\right) = \frac{3\pi}{4}$$

② First note that $\frac{2}{x^2+4} > 0$ for all $x \in \mathbb{R}$. Thus $\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx > 0$.

Next, as $\frac{2}{x^2+4}$ is continuous for all x, $\int_{-1}^{2} \frac{2}{x^2+4}\,dx$ exists (i.e. is finite).

So we are left to determine $\int_{-\infty}^{-1} \frac{2}{x^2+4}\,dx$. We compare this to

$$\int_{-\infty}^{-1} \frac{2}{x^2}\,dx = 2\int_{-\infty}^{-1} \frac{1}{x^2}\,dx \underset{\text{by symmetry}}{=} 2\int_1^{\infty} \frac{1}{x^2}\,dx.$$ We recognize this as a convergent p-integral, with $p=2$. Now note that $\frac{2}{x^2+4} \le \frac{2}{x^2}$ since $2x^2 \le 2x^2+8 \iff 0 \le 8$, holds for all reals. Thus $\int_{-\infty}^{-1} \frac{2}{x^2+4}\,dx \le \int_{-\infty}^{-1} \frac{2}{x^2}\,dx$. And so, by comparison, the given integral converges.

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$