Calculus II - Exam 3 - Fall 2013

November 14, 2013

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Avg: $\frac{47.6}{70}$

SD. 12.8

Directions: Upon completion of the examination and prior to its submission, please write and sign the Honor Code. **Justify** all answers/solutions. Make sure to indicate the test or theorem that you use. Calculators are not permitted, and all electronic devices should be off. Good luck!

1. [5 points] Give an example of a convergent geometric series with sum $\frac{8}{5}$. (The "tail" of the series should not consist of all zeros.)

The sum of a convergent geometric series with first term a and common ratio r is $\frac{a}{1-r}$. We wish this to equal $\frac{8}{5}$.

Thus $\frac{8}{5} = \frac{a}{1-r}$. We may choose any real number r so long as $|r|<1$, and then solve for a. A possible choice is $r = -1/2$; and so $\frac{8}{5} = \frac{a}{3/2} \Rightarrow a = \frac{16}{15}$. Thus, an example of such a convergent geometric series is $\sum_{n=0}^{\infty} \frac{16}{15}\left(\frac{1}{-2}\right)^n$

2. [5 points] State the Monotonic Sequence Theorem and give an example of such a sequence that the theorem describes. [5 points]

The Monotonic Sequence Theorem states that any bounded monotonic sequence converges. An example of such a sequence is $\{a_n\} = \{\frac{1}{n}\}$. Note[1] that the sequence is decreasing since $\frac{1}{n+1} < \frac{1}{n}$, and is bounded from below by 0, since $\frac{1}{n}$ is strictly positive.

3. [10 points] Use the Integral Test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

The given series converges if and only if the following indefinite integral converges:

$$\int_1^{\infty} x^2 e^{-x^3}\,dx.$$

Note that $x^2 e^{-x^3} > 0$ for all x, and that $x^2 e^{-x^3} = f(x)$ is a continuous function, and is decreasing.

So, we determine the nature of this indefinite integral.

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{b \to \infty} \int_1^b x^2 e^{-x^3}\,dx$$

To compute an antiderivative, we notice that the ~~integrand~~ is almost of the form $\int e^u\,du$, where $u = -x^3$ and $du = -3x^2$.

$$= \lim_{b \to \infty} -\frac{1}{3} \int_1^b -3x^2 e^{-x^3}\,dx$$

$$= \lim_{b \to \infty} -\frac{1}{3} \left(e^{-x^3} \right) \Big|_1^b = \lim_{b \to \infty} -\frac{1}{3} e^{-b^3} - \frac{-1}{3} e^{-1}$$

$$= \frac{1}{3e}.$$

So the integral, and thus the series, converges.

4. [10 points] Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^5}{4^n}$$

We use the Ratio Test to test for absolute convergence.

So we consider the following limit:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{(n+1)^5}{4^{n+1}}}{(-1)^n\dfrac{n^5}{4^n}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^5\cdot\frac{1}{4} = \frac{1}{4}\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^5$$

$$= \frac{1}{4}\cdot 1 = \frac{1}{4}.$$

As this limit is less than 1, by the Ratio Test the series is absolutely convergent.

Note that the Alternating Series Test can be used to establish convergence, but we cannot draw the conclusion of absolute convergence from it alone.

5. [10 points] Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

We start by using the Ratio Test. We consider the limit of the absolute value of the ratio of consecutive terms:

$$\lim_{n\to\infty} \left| \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \cdot \frac{5^n \sqrt{n}}{(2x-1)^n} \right| = \lim_{n\to\infty} \frac{1}{5} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot |2x-1|$$

$$= \frac{1}{5} |2x-1|$$

To have absolute convergence, we must have $\frac{1}{5}|2x-1| < 1$.

Thus, $|2x-1| < 5 \iff -5 < 2x-1 < 5 \iff -4 < 2x < 6$

$\iff -2 < x < 3$.

Thus, the radius of convergence is 2.5, as the series is centered at $a = 1/2$.

We test the endpoints of this interval since the Ratio Test is inconclusive here.

If $x = 3$, then the power series becomes $\sum_{n=1}^{\infty} \frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which is a divergent p-series with $p = 1/2 < 1$.

If $x = -2$, then the power series becomes $\sum_{n=1}^{\infty} \frac{-5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which is convergent (by an application of the Alternating Series Test).

Thus, the interval of convergence is $[-2, 3)$.

6. [10 points] Find the Taylor series for $1/x$ at $a = -3$. Further, what is the largest open interval for which we can say that the Taylor series obtained *represents* $1/x$?

Let the Taylor Series for this function be

$$c_0 + c_1(x+3) + c_2(x+3)^2 + c_3(x+3)^3 + \dots = \sum_{n=0}^{\infty} c_n (x+3)^n$$

We now find $c_0, c_1, \dots$

Let $f(x) = 1/x$

Then $f'(x) = \dfrac{-1}{x^2}$

$f''(x) = \dfrac{+2}{x^3}$

$f'''(x) = \dfrac{-3\cdot 2}{x^4}$

$\vdots$

$f^n(x) = \dfrac{(-1)^n n!}{x^{n+1}}$

$f(-3) = \dfrac{+1}{(-3)}$

$f'(-3) = \dfrac{-1}{(-3)^2}$

$f''(-3) = \dfrac{2}{(-3)^3}$

$f'''(-3) = \dfrac{-3\cdot 2}{(-3)^4}$

$\vdots$

$f^n(x) = \dfrac{(-1)^n n!}{(-3)^{n+1}}$

$c_0 = \dfrac{1/(-3)}{0!}$

$c_1 = \dfrac{-1/(-3)^2}{1!}$

$c_2 = \dfrac{2/(-3)^3}{2!}$

$c_3 = -\dfrac{3\cdot 2/(-3)^4}{3!}$

$\vdots$

$c_n = \dfrac{(-1)^n \cdot n!/(-3)^{n+1}}{n!}$

$= \dfrac{(-1)^n}{(-3)^{n+1}} = \dfrac{-1}{3^{n+1}}$

Thus the Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{-1}{(+3)^{n+1}} \cdot (x+3)^n$

An application of the Ratio Test will determine the open interval for which the series converges.

$$\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(x+3)^n} \right| = |x+3| \cdot \lim_{n\to\infty} \frac{1}{3} = \frac{|x+3|}{3}.$$

We need $\dfrac{|x+3|}{3} < 1 \Rightarrow (-6,0)$ is the largest open interval.

Note: We assume w/o proof that the Taylor series does converge to $1/x$ on this interval.

7. [5 points] Show how to use the Alternating Series Estimation Theorem to determine the number of terms one must take to estimate the sum of the Alternating Harmonic Series to within 0.005. (You needn't solve for n explicitly since you don't have a calculator and I don't want you spending time doing arithmetic.)

Recall the Alternating Harmonic Series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}$.

The Alternating Series Estimation Theorem - which can be applied here since we've already established this series converges - indicates that the n^{th} partial sum differs from the true sum by at most b_{n+1}.

Thus, we wish to solve the following inequality for n:

$$\frac{1}{n+1} \leq .005 \Rightarrow 200 \leq n+1 \Rightarrow 199 \leq n.$$

(The arithmetic on this wasn't hard at all.)

8. [5 points] The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

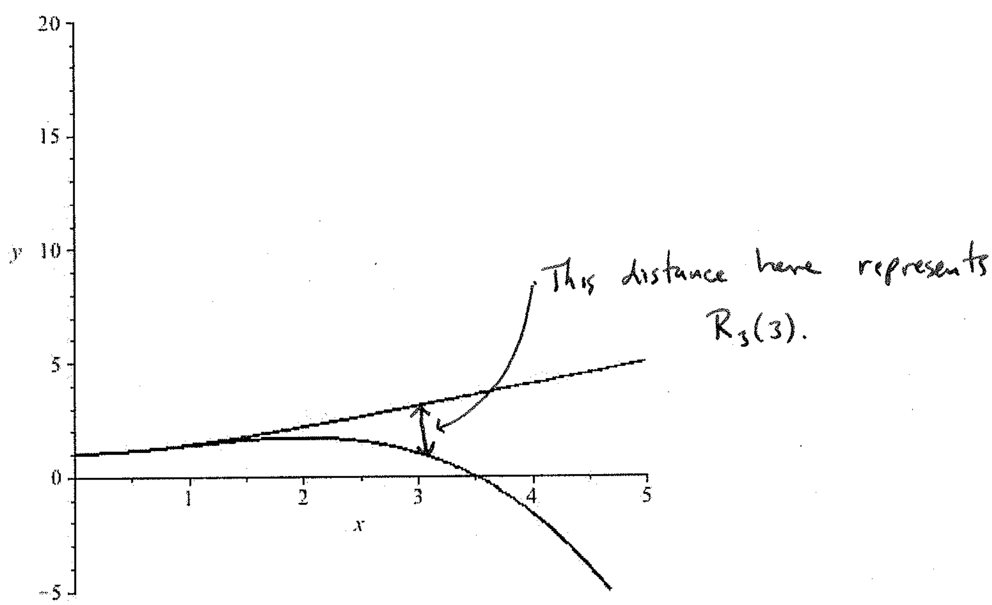
This is a p-series with $p = x$. We showed that p-series "output" a real number when $p > 1$. Thus, the domain of this function is $x > 1$.

9. [5 points] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series and no term of the series equals zero. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $a_n \to 0$ (i.e. the terms go to zero). If $a_n \to 0$, then $\left|\frac{1}{a_n}\right| \to \infty$. As the terms of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ do not go to zero, by the n^{th}-term test for divergence, this series diverges.

10. [5 points] In the figure below, I have plotted $f(x) = x + e^{-x}$ and its third-degree Taylor polynomial $T_3(x)$. The Maclaurin Series for this function is $1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{(n+1)!}$. Do the following three things: identify which curve is $T_3(x)$, compute $T_3(3)$ and use the figure to illustrate only the value of $R_3(3)$.

The graph of y=x+e^(-x) and its 3rd degree Taylor polynomial



Note that $T_3(x) = 1 + \dfrac{x^2}{2} - \dfrac{x^3}{6}$.

We may use ~~either of~~ the following facts to conclude that the bottom curve is $T_3(x)$: $x + e^{-x}$ is positive on $(0, \infty)$, ~~and~~ $T_3(x)$ is a cubic w/ negative leading coefficient, meaning that $T_3(x) \to -\infty$ as $x \to \infty$.

$T_3(3) = 1 + \dfrac{9}{2} - \dfrac{27}{6} = 1.$