Calculus II - Exam 3 - Fall 2016

November 17, 2016

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Total 68 pts.

Avg:

Directions: Upon completion of the examination and prior to its submission, please write and sign the Honor Code. **Justify** all answers/solutions. Make sure to indicate the test or theorem that you use. Calculators are not permitted, and all electronic devices should be off. Good luck!

1. [5 points] Theorem 6 of Chapter 11 states: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. The harmonic series shows that the converse to this statement is false, i.e. $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Give another example that shows that the converse is false.

   $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ is another example of a divergent series for which the terms go to zero. Note that $\lim_{n\to\infty} \ln(n) = \infty$ and so $\frac{1}{\ln(n)} \to 0$ as $n \to \infty$. Also as $\frac{1}{\ln(n)} > \frac{1}{n}$ for n large enough, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ diverges.

2. [5 points] For what values of r is the *sequence* $\{r^n\}$ convergent? For those values of r for which the sequence is convergent, state the limit.

   This is Fact $\boxed{9}$ on page 740:

   $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

   $$\lim r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

3. [5 points] State the Monotonic Sequence Theorem. Give an example of a sequence to which the theorem applies and prove that the hypotheses of the theorem apply to the example that you give.

The Monotonic Sequence Theorem states, every bounded monotonic sequence is convergent.

Claim $\{\frac{n-1}{n}\}$ is a bounded, monotonic sequence, and thus convergent.

Proof: Each term is bounded by 1 since $n > n-1$.

The sequence is monotonic since the $(n+1)^{st}$ term is greater than the n^{th} term; $a_{n+1} = \frac{n}{n+1} > \frac{n-1}{n} = a_n$ since

$$n^2 > n^2 - 1.$$

4. [5 points] What does the following geometric series converge to, $1 + 3/10 + 9/100 + 27/1,000 \ldots$?

The first term is $a=1$ and the common ratio is $r = 3/10$. The sum of a geometric series w/ first term a and common ratio r is $\frac{a}{1-r}$.

Thus, this geometric series converges to

$$\frac{1}{1 - 3/10} = \frac{1}{7/10} = \frac{10}{7}$$

5. [8 points each] **Complete four of the following five. Indicate which problem you omit with a slash through it.** For each of the following series, determine whether or not the series converges. If the series contains any negative terms, please test for absolute convergence. State the theorem (i.e. test) that you use to draw your conclusion.

(a)

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

We'll use the integral test, and so we consider the following integral $\lim_{b\to\infty} \int_2^b \frac{\ln x}{x^2}\,dx$. Note that $\frac{\ln x}{x^2} > 0$ for $x \geq 2$ and by ~~an~~ applying L'Hopital's Rule we see that $\lim_{x\to\infty} \frac{\ln x}{x^2} = 0$.

We find an antiderivative using integration by parts, with $u = \ln x$, $du = \frac{1}{x}dx$, $dv = \frac{dx}{x^2}$, $v = \frac{-1}{x}$.

Thus, $\int \frac{\ln x\,dx}{x^2} = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$

So, $$\lim_{b\to\infty} \int_2^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Bigg|_2^b$$

$$= \lim_{b\to\infty} \left(-\frac{\ln(b)}{b} - \frac{1}{b}\right) - \left(-\frac{\ln(2)}{2} - \frac{1}{2}\right)$$

$$= \lim_{b\to\infty} \frac{-1/b}{1} - 0 + \frac{\ln 2}{2} + \frac{1}{2}$$

$$= 0 - 0 + \frac{\ln 2}{2} + \frac{1}{2}.$$

This is a finite number. Thus the integral converges, and by the Integral Test the series does as well.

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n^3+n+1}}$$

Let us examine the terms in the series:

$$\frac{\sqrt[3]{n}}{\sqrt{n^3+n+1}} \leq \frac{\sqrt[3]{n}}{\sqrt{n^3}} = \frac{n^{1/3}}{n^{3/2}} = \frac{1}{n^{7/6}}$$

So the given series is less than the convergent p-series

$\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$, and thus it converges.

(c)

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$$

(Hint: re-express the n^{th}-term of the series before applying any test.)

Note that $\cos(n\pi)$ equals -1 when n is odd and equals $+1$ when n is even. So we may rewrite $\cos(\pi n)$ as $(-1)^n$. That is, the series is $\sum_{n=1}^{\infty} \frac{n(-1)^n}{2^n}$. As this series contains negative terms, according to the directions, we shall test for absolute convergence. We'll use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)(-1)^{n+1}}{2^{n+1}}}{\frac{n(-1)^n}{2^n}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{2} = \frac{1}{2}.$$

As this limit is less than 1, the given series converges absolutely. And, as absolute convergence implies convergence, we know the given series converges.

(d)

$$\sum_{n=1}^{\infty} \left(\frac{n}{\ln n}\right)^n$$

typo! Should be $n=2$, since $\ln(1)=0$ and so the first term would be undefined.

We'll apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{\ln n}\right)^n} = \lim_{n\to\infty} \frac{n}{\ln n}$$

and we arrive at an indeterminate form of $\frac{\infty}{\infty}$. We apply L'Hopital's Rule to see that we have $\lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$.

Thus, by the Root Test, the given series is divergent.

(e)

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$$

We'll apply the Ratio Test. So, consider

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(2n+3)!}}{\frac{(-3)^n}{(2n+1)!}} \right| = 3 \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 3 \cdot 0 = 0.$$

So, by the Ratio Test, we have absolute convergence and thus convergence.

6. [8 points] Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10.$$

The left side is a geometric series with first term $a = e^{0 \cdot c} = 1$ and common ratio e^c. As we know the series converges, we know its sum is both 10 and $\frac{a}{1-r} = \frac{1}{1-e^c}$.

We equate and solve for c:

$$10 = \frac{1}{1-e^c}$$

$$\frac{1}{10} = 1 - e^c$$

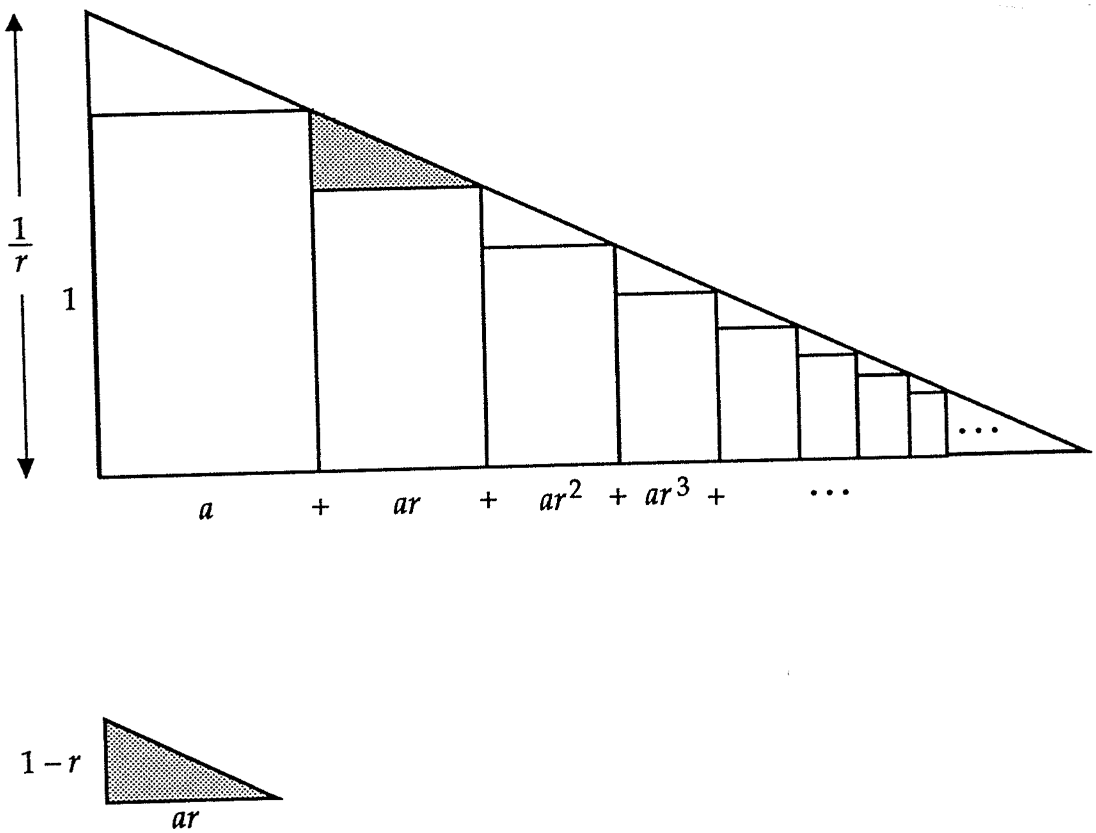
$$e^c = \frac{9}{10}$$

$$c = \ln(9/10).$$

7. [8 points] Explain how the figure given proves the provided equations. [R. Nelsen, Proofs without words]

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\frac{a + ar + ar^2 + ar^3 + \cdots}{1/r} = \frac{ar}{1-r}$$



The large triangle and the small (shaded) triangle are similar right triangles. Thus, the ratio of the
9 legs of the large triangle equals the ratio of the corresponding legs of the small triangle. That is,

$$\frac{a + ar + ar^2 + \cdots}{1/r} = \frac{ar}{1-r}.$$

One obtains the first equation from this after dividing thru by r.