Calculus II - Exam 3 - Spring 2013

April 25, 2013

Name: Key.

Honor Code Statement: I have neither given nor received any unauthorized aid on this exam.

Additional Statement: I have not observed another violating the Honor Code.

Signature:

Total 90 points

Directions: Justify all answers/solutions. Calculators, notes and texts are not permitted. The point value of each problem is indicated in brackets. When you use a particular theorem or test, please **give the name** of the result you are using. Good luck!

1. [4] Define what it means for a sequence to be **bounded**.

A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq M$ $\forall n \geq 1$.

A sequence $\{a_n\}$ is bounded below if there is a number m such that $m \leq a_n$ $\forall n \geq 1$.

If the sequence $\{a_n\}$ is bounded above and below, then it is bounded.

2. [6] Determine whether the given sequence is increasing, decreasing, or not monotonic. Is the sequence bounded? Justify your answers!

$$a_n = \frac{1}{5^n}$$

The sequence is decreasing since $\frac{1}{5^{n+1}} \leq \frac{1}{5^n}$ for all $n \geq 1$.

The sequence is bounded since $0 \leq \frac{1}{5^n} \leq 1$ $\forall n \geq 1$.

Therefore by the Monotonic Sequence Theorem this sequence is convergent

All further questions are [10] points each.

3. Determine whether the series is convergent. If it is convergent find its sum.

(i) $1 + 0.4 + 0.16 + 0.064 + ...$

This is a geometric series (assumingly) with first term $a=1$ and common ratio $r=0.4$. Thus, we know that as $|r|<1$, the sum is $\frac{a}{1-r} = \frac{1}{1-0.4} = \frac{1}{6/10} = \frac{10}{6} = \frac{5}{3}$

(ii) $\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^n} = \sum_{n=1}^{\infty} \left(\frac{-6}{5}\right)^{n-1} \cdot \frac{1}{5}$

This too is a geometric series. However, as $r = \frac{-6}{5}$ is larger than 1 in absolute value, we know the series diverges.

4. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{7n - n^{1/3}}{n^5}$$

Let us write this as $\sum_{n=1}^{\infty} \frac{7n}{n^5} - \sum_{n=1}^{\infty} \frac{n^{1/3}}{n^5}$

$$= 7\sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{n^{14/3}}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{14/3}}$ are convergent p-series, the given series is convergent.

5. Use the integral test to determine if the following ~~sequence~~ series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

By the Integral Test, the behavior of $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is the same as $\int_2^{\infty} \frac{1}{x^2-1}\,dx$, so we compute this integral.

To find an antiderivative, we ~~do~~ find a partial fraction decomposition

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow$$

$1 = A(x-1) + B(x+1)$

when $x=1$ $\quad B = \frac{1}{2}$

$x=-1$ $\quad A = -1/2$

Thus, $$\int_2^{\infty} \frac{1}{x^2-1}\,dx = \frac{1}{2}\int_2^{\infty} \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\lim_{b\to\infty} \int_2^b \frac{1}{x-1} - \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln|x-1| - \ln|x+1| \Big|_2^b$$

$$= \frac{1}{2}\lim_{b\to\infty} \ln\left|\frac{x-1}{x+1}\right| \Big|_2^b = \frac{1}{2}\lim_{b\to\infty} \ln\frac{(b-1)}{(b+1)} - \ln(1/3)$$

$$= \frac{1}{2}(-\ln 1/3) = \frac{\ln 3}{2}.$$

As this integral converges, the series converges.

6. Test the series for convergence.

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$$

We use the Alternating Series Test, as $b_n = \frac{2n}{4n^2+1} > 0$ for all n so the terms of the series truly alternate.

(1) We show the terms are decreasing by showing that the first derivative of the function $f(x) = \frac{2x}{4x^2+1}$ is negative for x big enough.

$$f'(x) = \frac{(4x^2+1)\cdot 2 - 2x\cdot 8x}{(4x^2+1)^2}$$ by the quotient rule.

This is negative when the numerator is negative, i.e. when

$$8x^2+2-16x^2 < 0$$

$$\Leftrightarrow \quad -8x^2+2 < 0 \quad \Leftrightarrow \quad \frac{1}{4} < x^2$$

So $f(x)$ is decreasing on $(\frac{1}{2}, \infty)$.

Thus the terms are decreasing.

(2) Now we show the terms go to zero:

$$\lim_{n\to\infty}\frac{2n}{4n^2+1} = \lim_{n\to\infty}\frac{\frac{2}{n}}{4+\frac{1}{n^2}} = 0.$$

Thus the given series is convergent by the Alternating Series Test.

7. Test for absolute convergence.

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

We use the ratio test.

$$\lim_{n \to \infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}} \right| = \lim_{n \to \infty} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{3}{n+1} = 0.$$

Thus as $0 < 1$, the ratio test says that the series must converge absolutely.

8. Find the radius of convergence and interval of convergence.

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

Let $a_n = \dfrac{x^n}{n\,3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{x^{n+1}}{(n+1)3^{n+1}}}{\dfrac{x^n}{n\,3^n}}\right| = \left|\frac{x}{3}\cdot\frac{n}{n+1}\right| \to \left|\frac{x}{3}\right| \text{ as } n\to\infty.$$

Thus, by the Ratio Test the given series converges if $\left|\frac{x}{3}\right| < 1$ and diverges if $\left|\frac{x}{3}\right| > 1$. Thus it converges if $-3 < x < 3$ and diverges if $x < -3$ or $x > 3$. We now test for convergence at $x = 3$ and $x = -3$.

At $x = 3$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which we recognize as the divergent harmonic series.

At $x = -3$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which we recognize as the convergent alternating harmonic series.

Thus the radius of convergence is 3 and the interval of convergence is $[-3, 3)$.

9. Find a power series representation for the function.

$$f(x) = \frac{1}{1+9x^2}$$

If we rewrite $f(x)$ as $\frac{1}{1-(-9x^2)}$, then we recognize this as the sum of a geometric series with $a=1$ and $r=-9x^2$.

Thus, we must have $|-9x^2|<1$, that is $9x^2<1$, or

$x^2 < \frac{1}{9}$ or $|x| < \frac{1}{3}$.

And the representation is $\sum_{n=0}^{\infty} (-9x^2)^n = \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}$

$= 1 - 9x^2 + 81x^4 - 729x^6 \ldots$

10. Find the Taylor series for $f(x)$ (assuming that it has one) centered at the value $a = 2$

$$f(x) = \ln(x)$$

We use the method of Taylor.

f(x) and its derivatives	these evaluated at a=2	the coefficients
$f(x) = \ln x$	$f(2) = \ln 2$	$c_0 = \frac{\ln 2}{0!}$
$f'(x) = \frac{1}{x}$	$f'(2) = \frac{1}{2}$	$c_1 = \frac{1/2}{1!}$
$f''(x) = \frac{-1}{x^2}$	$f''(2) = \frac{-1}{4}$	$c_2 = \frac{-1/4}{2!}$
$f'''(x) = \frac{+2}{x^3}$	$f'''(2) = \frac{2}{2^3}$	$c_3 = \frac{2}{2^3 \cdot 3!}$
$f^4(x) = \frac{-3\cdot 2}{x^4}$	$f^4(2) = \frac{-3\cdot 2}{2^4}$	$\vdots$
$\vdots$	$\vdots$	
$f^n(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$	$f^n(2) = (-1)^{n-1}\frac{(n-1)!}{2^n}$	$c_n = \frac{(-1)^n (n-1)!}{2^n \cdot n!} = \frac{(-1)^n}{2^n \cdot n}$

Thus, the Taylor series is $\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n \cdot n}(x-2)^n$