Name Solution Key

ID number

Sections **B**

Calculus II (Math 122) Final Exam, 11 December 2013

This is a closed book exam. Notes and calculators are not allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, ln(), exp(), fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the Honor Code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

1	10
2	5
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10
Total	95

Honor Code: I have neither given nor received unauthorized aid on this exam.

Signature:

1. [10 points] Below is a sketch of the direction field for a differential equation. Sketch the graphs of the solutions that satisfy the given initial conditions $y(4) = 1$ and $y(-6) = 3$.

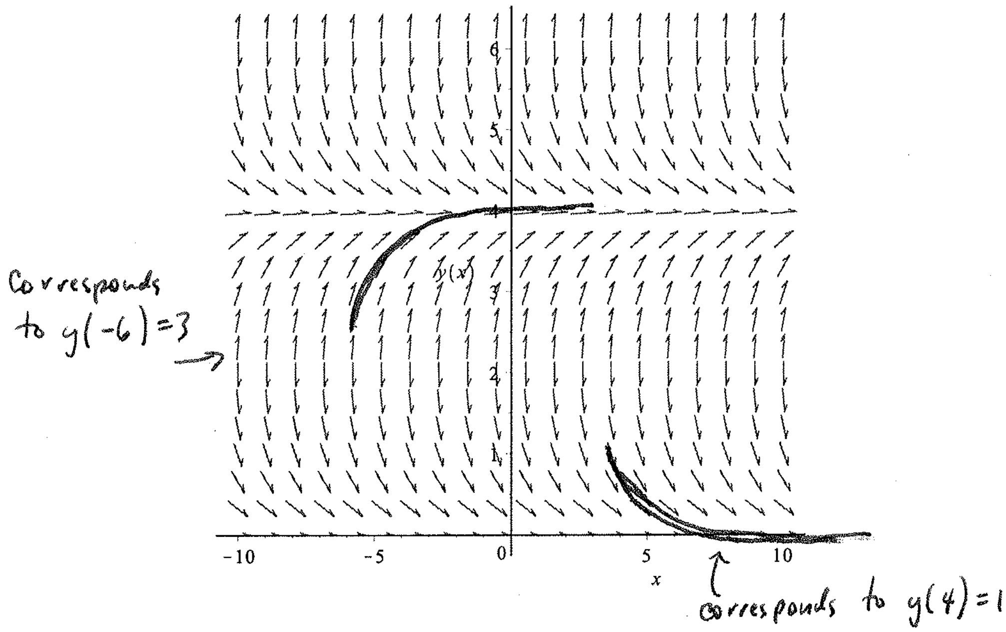


Figure 1: The direction field for $y' = -\tan(\frac{\pi y}{4})$

Now find **all** the equilibrium solutions (not just those seen in the sketch).

We see that $y=4$ is an equilibrium solution from the figure.

If we set $y' = 0$, we find the others. Thus, $0 = -\tan\left(\frac{\pi y}{4}\right)$

$\Leftrightarrow 0 = \tan\left(\frac{\pi y}{4}\right) \Leftrightarrow \tan^{-1}(0) = \frac{\pi y}{4} \Rightarrow \tan^{-1}(0) = k\pi$ for $k \in \mathbb{Z}$.

Thus $k\pi = \frac{\pi y}{4} \Rightarrow y = 4k$, $k \in \mathbb{Z}$. The equilibrium solutions occur at $y = 0, \pm 4, \pm 8, \ldots$

2. [5 points] Consider the logistic differential equation $\frac{dP}{dt} = rP(M-P)$ and recall that the second derivative of P is given by $r^2P(2P-M)(P-M)$. Assume that as you collect data on a population P over time t, you notice that the data fits the logistic model and you notice that the rate of change of the population slows when the population hits 5,000. What might you predict about the carrying capacity M of the population your are observing? Justify your answer.

A slow down in the rate of change occurs when the 2nd derivative of P turns from positive to negative. From the given information about the 2nd derivative, we identify where the 2nd derivative is 0:

at $P=0$, at $2P-M=0 \Leftrightarrow P=\frac{M}{2}$, and at $P=M$.

Our focus is drawn to $P=\frac{M}{2}$: when P is less than $\frac{M}{2}$ the 2nd derivative is positive and when P is greater than $\frac{M}{2}$ the 2nd derivative is negative. Thus, we can conclude that 5,000 is half of M and so we'd expect the population to reach 10,000.

3. [5 points each] Determine whether the following series converges or diverges. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

(a) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

We use the Ratio Test, and so we consider $\lim_{n\to\infty} \dfrac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \lim_{n\to\infty} \frac{n+1}{100} = \infty.$

Thus, the series is divergent.

You might also use the n^{th} term test for divergence and show that $\frac{n!}{100^n} \to \infty$ as $n \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n \sin^2(n)}{1+n^3}$

We will make use of the Direct Comparison Test.

Note that $0 \leq \sin^2 n \leq 1$. Thus, we have

$0 \leq \frac{n \sin^2(n)}{1+n^3} \leq \frac{n}{1+n^3}$. Now note $\frac{n}{1+n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$ $\forall n \geq 1$.

Thus, since all terms in the given series are non-negative, the sequence of partial sums is monotonic and bounded by $\sum \frac{1}{n^2}$, which is a convergent p-series. Thus, the given series is convergent.

4. [10 points] Solve the following separable differential equation.

$$\frac{dy}{dx} = \frac{x(y^2+1)}{x+1}.$$

To solve this differential equation, we may solve this integral equation:

$$\int \frac{dy}{y^2+1} = \int \frac{x}{x+1}\, dx$$

We recognize the left side as $\tan^{-1}(y) + C$.

We manipulate the integrand on the right side (via polynomial long division, if you wish) to obtain

$$\tan^{-1}(y) + C = \int 1 - \frac{1}{x+1}\, dx$$

$$\tan^{-1}(y) + C = x - \ln|x+1|$$

$$\Rightarrow \quad \tan^{-1}(y) = x - \ln|x+1| + \tilde{C}$$

$$y = \tan\left(x - \ln|x+1| + \tilde{C}\right)$$

5. [10 points] Evaluate $\int xe^{-x^3}\,dx$ as an infinite series.

Recall that the Maclaurin Series for e^x is $1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

By substitution e^{-x^3} has Maclaurin Series $1 + \frac{(-x^3)}{1!} + \frac{(-x^3)^2}{2!} + \dots$

$= 1 - \frac{x^3}{1!} + \frac{x^6}{2!} - \frac{x^9}{3!} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$.

So the integrand can be written as $x \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{n!}$.

The integral becomes $\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{3n+2}}{3n+2} + C$.

Give an approximation of $\int_0^1 xe^{-x^3}\,dx$ **and** bound the error of your approximation.

From the above, and FTC Part 2 $\int_0^1 x e^{-x^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{3n+2}$.

Thus, the definite integral is the sum of a convergent alternating series. We can approximate the sum via a partial sum, let's say the partial sum $S_4 = \frac{1}{2} - \frac{1}{5} + \frac{1}{16} - \frac{1}{66}$.

By the Alternating Series Estimation Theorem, this estimate is "off" by no more than $\frac{(-1)^4}{4!} \cdot \frac{1}{3 \cdot 4 + 2} = \frac{1}{(24)(14)}$.

6. [10 points] A tank contains 100 gallons of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find an expression for the amount of fertilizer F in the tank at time t.

The rate in of fertilizer is: $\frac{1 \text{lb}}{\text{gal}} \cdot \frac{1 \text{gal}}{\text{min}} = \frac{1 \text{ lb.}}{\text{min}}$.

The rate out of fertilizer is: $\frac{F(t)}{100-2t} \cdot \frac{3 \text{gal}}{\text{min}} = \frac{3F}{100-2t}$.

Thus, $\frac{dF}{dt} = 1 - \frac{3F}{100-2t}$. This is a first-order linear differential equation, which we write in standard form as.

$\frac{dF}{dt} + \frac{3}{100-2t} F = 1$. To solve it (and so find F in terms of t), we multiply through by an integrating factor

$$I(t) = e^{\int \frac{3}{100-2t} dt} = e^{-\frac{3}{2}\int \frac{-2\,dt}{100-2t}} = e^{-3/2 \ln|100-2t|} = (100-2t)^{-3/2}$$

So our diff. eq. becomes:

$$(100-2t)^{-3/2} \frac{dF}{dt} + 3(100-2t)^{-5/2} F = (100-2t)^{-3/2}$$

$$\frac{d\left(F \cdot (100-2t)^{-3/2}\right)}{dt} = (100-2t)^{-3/2}$$

We integrate both sides with respect to t, applying the FTC on the right simultaneously to obtain

$$F \cdot (100-2t)^{-3/2} = \int \frac{1}{(100-2t)^{3/2}}\, dt$$

Let $u = 100-2t$

$du = -2\, dt$

$$= -\frac{1}{2} \int \frac{-2\, dt}{(100-2t)^{3/2}}$$

$$= -\frac{1}{2} \int u^{-3/2}\, du$$

$$= -\frac{1}{2} \cdot -2\, u^{-1/2} + C$$

$$F \cdot (100-2t)^{-3/2} = (100-2t)^{-1/2} + C$$

$$F = (100-2t)^{1} + \frac{C}{(100-2t)^{-3/2}} = (100-2t) + C(100-2t)^{3/2}$$

We find C from the given, $F(0) = 0$.

$$0 = (100-0) + C(100-0)^{3/2}$$

$$0 = 100 + C \cdot 1000$$

$$-\frac{1}{10} = C$$

Thus, $F(t) = (100-2t) - \frac{1}{10}(100-2t)^{3/2}$ for $0 \le t \le \frac{100}{3}$

7. [10 points] Demonstrate 3 iterations in Euler's method with step size of $h = 0.5$ for the differential equation and initial condition given.

$$y' = 1 + y, \quad y(0) = 1$$

We are given $x_0 = 0$, $y_0 = 1$ and asked to use step size $h=0.5$.

$y_1 = y_0 + h\,F(x_0, y_0) = 1 + \frac{1}{2}(1+1) = 2 \Rightarrow x_1 = 1/2,\ y_1 = 2$

$y_2 = y_1 + h\,F(x_1, y_1) = 2 + \frac{1}{2}(1+2) = 3.5 \Rightarrow x_2 = 1,\ y_2 = 3.5$

$y_3 = y_2 + h\,F(x_2, y_2) = 3.5 + \frac{1}{2}(1+3.5) = 5.75 \Rightarrow x_3 = 1.5,\ y_3 = 5.75$.

8. [10 points] Solve the following differential equation: $4x^3y + x^4y' = \sin^3(x)$.

This is a first-order linear differential equation, with the left side already in a convenient form. The differential equation may be rewritten as

$$\frac{d(x^4 \cdot y)}{dx} = \sin^3(x).$$

We now integrate both sides w.r.t. x:

$$\int \frac{d(x^4 \cdot y)}{dx}\, dx = \int \sin^3(x)\, dx$$

By FTC we obtain

$$x^4 \cdot y = \int \sin^3 x \, dx.$$

$$= \int \sin x \, (1 - \cos^2 x)\, dx$$

Let $u = \cos x$
$du = -\sin x$

$$= -\int -\sin x \, (1 - \cos^2 x)\, dx$$

$$= -\int (1 - u^2)\, du$$

$$= -u + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$

$$\Rightarrow y = \frac{\cos^3 x}{3x^4} - \frac{\cos x}{x^4} + \frac{C}{x^4}$$

9. [10 points] Find an equation of the tangent to the curve given by $x = \sin^3(\theta),\ y = \cos^3(\theta)$ at $\theta = \pi/6$.

We first find the slope of the tangent using Formula 1, as found on page 669 of the text.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos^2\theta \cdot (-\sin\theta)}{3\sin^2\theta(\cos\theta)} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta.$$

At the point of interest the slope is $-\cot(\pi/6) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$

When $\theta = \pi/6$, $x = \sin^3(\pi/6) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and $y = \cos^3(\pi/6) = \left(\frac{\sqrt{3}}{2}\right)^3$
$= \frac{3\sqrt{3}}{8}$

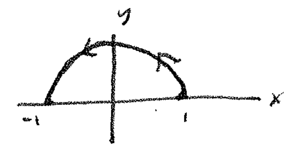
Using the point-slope form of a line, we obtain

$$y - \frac{3}{8}\sqrt{3} = -\sqrt{3}\left(x - \frac{1}{8}\right)$$

or
$$y = -\sqrt{3}\,x + \frac{\sqrt{3}}{8} + \frac{3}{8}\sqrt{3}$$

$$y = -\sqrt{3}\,x + \frac{\sqrt{3}}{2}$$

10. [10 points] **Length is independent of parametrization!** The following exercise is meant to demonstrate that the length of a curve is independent of the parametrization given to it (so long as we don't "double-back"). Below are two different parameterizations of the unit semi-circle. Show that each provides arc length of π.



(1) $x = \cos(2t), \quad y = \sin(2t), \quad 0 \le t \le \pi/2$

(2) $x = \sin(\pi t), \quad y = \cos(\pi t), \quad -1/2 \le t \le 1/2$

We know by Theorem 5 of Section 10.2 that the arc length for these parametric equations is given by $L = \displaystyle\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

We apply this formula to each parametrization:

(1)

$$L = \int_0^{\pi/2} \sqrt{4\sin^2(2t) + 4\cos^2(2t)}\, dt$$

$$= \int_0^{\pi/2} 2\, dt$$

$$= 2t\Big|_0^{\pi/2} = 2 \cdot \frac{\pi}{2} = \pi$$

(2)

$$L = \int_{-1/2}^{1/2} \sqrt{\pi^2\cos^2(\pi t) + \pi^2\sin^2(\pi t)}\, dt$$

$$= \int_{-1/2}^{1/2} \pi\, dt$$

$$= \pi t\Big|_{-1/2}^{1/2} = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) = \pi.$$

So we see that in either case the length is π.

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$