Name Solution Key.

ID number

Sections **C**

Calculus II (MATH 0122) Final Exam, 14 December 2016

This is a closed book exam. Notes, calculators, cell-phones are not allowed – the only allowable items are pens, pencils and erasers. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, ln(), exp(), fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the Honor Code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

Please complete seven of the eight questions. Indicate which you will omit by putting a slash through that question.

1	
2	
3	
4	
5	
6	
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8	
Total	

Best score: 70 points

Average: 60 points (wow!)

Honor Code:

I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.

Signature:

1. [10 points] Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$$

Let $a_n = \dfrac{(-1)^n 4^n}{\sqrt{n}} x^n$. To determine convergence, we use the Ratio Test. Consider, $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \dfrac{\left|\dfrac{(-1)^{n+1}4^{n+1}x^{n+1}}{\sqrt{n+1}}\right|}{\left|\dfrac{(-1)^n 4^n x^n}{\sqrt{n}}\right|}$

$$= \lim_{n\to\infty} 4\frac{\sqrt{n}}{\sqrt{n+1}}|x| = 4|x| \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}$$

$$= 4|x|.$$

By the Ratio Test we have convergence when $4|x| < 1 \iff |x| < 1/4$

i.e. convergence for $-1/4 < x < 1/4$. And, so the radius of convergence is $1/4$.

The Ratio Test is inconclusive at the endpoints of this interval, so we test these separately.

At $x = 1/4$, the series equals $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test.

At $x = -1/4$ the series equals $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges since it is a p-series ~~p~~ w/ $p = 1/2 \leq 1$.

We find that the interval of convergence is $\left(-1/4, 1/4\right]$

2. [10 points] Find a power series representation for the function by first using partial fractions and determine the interval of convergence.

$$f(x) = \frac{2x+3}{x^2+3x+2}$$

Note $f(x) = \dfrac{2x+3}{(x+2)(x+1)}$. We do a partial fraction decomposition.

$$\frac{2x+3}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

So, $2x+3 = A(x+1) + B(x+2)$

When $x=-1$, we get $1 = B$

When $x=-2$, we get $-1 = -A \Rightarrow A=1$

Thus, $f(x) = \dfrac{1}{x+2} + \dfrac{1}{x+1}$.

We re-write these as

$$f(x) = \frac{1}{2-(-x)} + \frac{1}{1-(-x)}$$

$$= \frac{1}{2\left(1-\left(\frac{-x}{2}\right)\right)} + \frac{1}{1-(-x)}$$

Each of these parts may be viewed as the sum of a geometric series.

The first part is a geometric series w/ first term $a = 1/2$ and ratio $r = \frac{-x}{2}$

The second part is a geometric series w/ first term $a=1$ and ratio $r = -x$.

The first may be written as $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{-x}{2}\right)^{n-1}$, w/ interval of convergence $\left|\frac{-x}{2}\right| < 1$, i.e. $(-2,2)$

The second may be written as $\sum_{n=1}^{\infty} 1\,(-x)^{n-1}$, w/ interval of convergence $(1,1)$.

Thus, $f(x) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\left(\frac{-x}{2}\right)^{n-1} + (-x)^{n-1}\right)$ when $x \in (1,1)$, where the interval is the narrower/smaller of the two.

3. [10 points] Given that $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, evaluate the indefinite integral as a power series, what is the radius of convergence? (Caution: please note that the argument for the natural logarithm function given above is different to the argument when this function appears as part of the integrand below.)

$$\int x^2 \ln(1+x^2)dx$$

By a substitution, we have that the power series for

$\ln(1+x^2)$ is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}$

Thus, $x^2 \ln(1+x^2)$ may be represented as $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+2}}{n}$

We may now approach the integral problem as

$$\int \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+2}}{n} dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{x^{2n+3}}{(2n+3)}$$

The radius of convergence for $\ln(1+x)$ is 1, and so the same is true for the obtained series.

4. [10 points] Find the Taylor series for $f(x) = \sin(x)$ centered at the value of π. [Assume that $f(x)$ has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the radius of convergence.

We begin by finding the derivatives of $f(x)$.

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$

and then this pattern continues.

Also, $f(\pi) = 0$, $f'(\pi) = -1$, $f''(\pi) = 0$, $f'''(\pi) = 1$

and then this pattern continues.

So the coefficients for the Taylor series are:

$$c_0 = \frac{0}{0!} , \quad c_1 = \frac{-1}{1!} , \quad c_2 = \frac{0}{2!} , \quad c_3 = \frac{1}{3!}$$

and in general $c_{2n} = 0$, $c_{4n+1} = \dfrac{(-1)}{(4n+1)!}$, $c_{4n+3} = \dfrac{1}{(4n+3)!}$

Thus, the Taylor Series is:

$$\frac{-1}{1!}(x-\pi)^1 + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!}$$

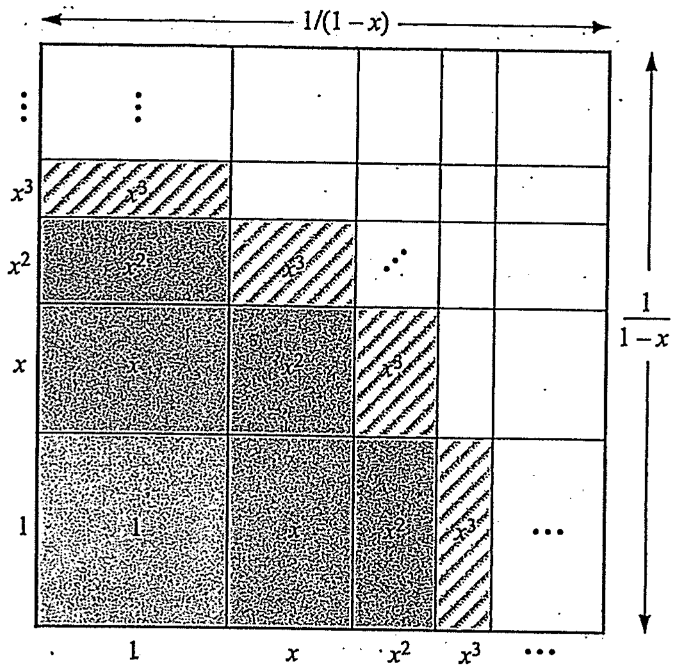
We can compute the radius of convergence using the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-\pi)^{4n+3}}{(4n+3)!} \cdot \frac{(4n+1)!}{(x-\pi)^{4n+1}} \right| = (x-\pi)^2 \lim_{n\to\infty} \frac{1}{(4n+3)(4n+2)} = 0$$

Thus, the series converges for all x, i.e. the radius of convergence is infinite.

5. [10 points] [R. Nelsen] Consider the figure below. Use the figure to explain the following implication:

$$x \in [0,1) \Rightarrow 1 + 2x + 3x^{2} + 4x^3 + \cdots = \left(\frac{1}{1-x}\right)^2$$



$$x \in [0, 1) \quad \Rightarrow \quad 1 + 2x + 3x^2 + 4x^3 + \cdots = \left(\frac{1}{1-x}\right)^2.$$

We know that for $x \in [0,1)$, we have $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.

We see in the figure a square whose side is $\frac{1}{1-x}$. Its area is thus $\left(\frac{1}{1-x}\right)^2$. At the same time we may find the area by using the smaller rectangles w/in the large square. There is ~~one~~ 1 rectangle of area 1, 2 rectangles of area x, 3 rectangles of area x^2, and so on. Thus the area is $1 + 2x + 3x^2 + 4x^3 + \cdots$.

We have computed the same value in two different ways, so these must be equal.

6. [10 points] Consider the following differential equation

$$\frac{dv}{dt} = -v[v^2 - (1+a)v + a],$$

where a is a positive constant such that $0 < a < 1$.

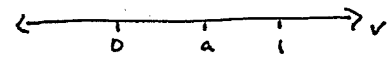
(a) For what values of v is $\frac{dv}{dt} = 0$?

Notice that we can factor the right side of the above equation to obtain

$$\frac{dv}{dt} = -v(v-a)(v-1).$$ (This allows for easier analysis.)

Thus, $\frac{dv}{dt} = 0$ when $v = 0$, $v = a$ or $v = 1$.

Perhaps useful to have this illustration



(b) For what values of v is v increasing?

With $0 < a < 1$, $\frac{dv}{dt}$ is ~~negative~~ positive when $v < 0$ or $a < v < 1$,

and this corresponds to where v is increasing.

(c) For what values of v is v decreasing?

$\frac{dv}{dt}$ is ~~decreasing~~ negative when $0 < v < a$ and $v > 1$

and so this corresponds to where v is decreasing.

7. [10 points] Use Euler's method with step size 0.5 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = x^2y - \frac{1}{2}y^2$, $y(0) = 1$. (That is, find (x_2, y_2).)

We are given that $(x_0, y_0) = (0, 1)$

and step size is $h = 0.5 = 1/2$. Thus, $x_1 = 1/2$, $x_2 = 1$.

We wish to find $y(1)$ or rather y_2.

$$y_1 = y_0 + h\,F(x_0, y_0) \qquad \text{where } F(x,y) = x^2y - \tfrac{1}{2}y^2$$

$$y_1 = 1 + \tfrac{1}{2}\left(0 - \tfrac{1}{2}\right) = 1 - 1/4 = 3/4$$

Then, $y_2 = y_1 + h\,F(x_1, y_1)$

$$y_2 = 3/4 + \tfrac{1}{2}\left(\tfrac{1}{4} \cdot \tfrac{3}{4} - \tfrac{1}{2}\,\tfrac{9}{16}\right)$$

$$= 3/4 + \tfrac{1}{2}\left(3/16 - 9/32\right)$$

$$= 3/4 + 3/32 - 9/64 = \frac{45}{64}.$$

Thus, $(x_2, y_2) = (1, 45/64)$.

8. [10 points] Solve the following separable differential equation.

$$\frac{dy}{dx} = \frac{x\sin(x)}{y}, \quad y(0) = -1$$

We note that this is a separable differential equation.

Thus, to solve for y, we may write

$$\int y \, dy = \int x \sin x \, dx$$

The left side is easy to integrate, whereas the right will require integration by parts, and for this we choose $u = x$, $dv = \sin x \, dx$, where $du = 1 \, dx$, $v = -\cos x$. Thus,

$$\frac{y^2}{2} = -x\cos x - \int -\cos x \, dx$$

$$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

If $x=0$, $y=-1$ (by the given) and so we may solve for C.

$$\frac{(-1)^2}{2} = 0 + \sin(0) + C \implies C = \frac{1}{2}$$

Thus,

$$\frac{y^2}{2} = -x\cos x + \sin x + \frac{1}{2}$$

$$\implies y^2 = -2x\cos x + 2\sin x + 1$$

$$\implies y = -\sqrt{-2x\cos x + 2\sin x + 1}$$

Note we take negative square root since $y(0)$ is negative

Trigonometric Identities

Addition and subtraction formulas

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Double-angle formulas

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$

Half-angle formulas

- $\sin^2 x = \frac{1 - \cos(2x)}{2}$
- $\cos^2 x = \frac{1 + \cos(2x)}{2}$

Others

- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$