Name Solutions
ID number
Sections **C and D**

Calculus II (Math 122) Final Exam, 19 May 2012

This is a closed book exam. No notes or calculators are allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, ln(), exp(), fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the honor code, (I have neither given nor received unauthorized aid on this exam) in the space provided below. Please remember that you are also obligated to report violations of the honor code. Good luck!

1	9
2	12
3	27
4	10
5	25
6	32
7	15
8	10
Total	140

Honor Code: I have neither given nor received unauthorized aid on this exam.

Signature:

1. [3 each] State clearly and precisely the following:

(a) Definition of an infinite sequence (using the language of functions)

An infinite sequence is a function with the domain of the positive integers.

An example is: 1, 2, 4, 8, 16, ...

(b) Definition of a geometric series

A geometric series is a series of the form $a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$.

In other words, it is a series in which the ratio of each consecutive pair is constant r.

An example is: $1 + 3 + 9 + 27 + 81 + \ldots = \sum_{n=1}^{\infty} 3^{n-1}$

(c) Definition of a continuous function, f(x), at x=c where c is in the domain of f

A function f is continuous at c if

$$f(c) = \lim_{x \to c} f(x).$$

2. [2 points each] **Fill in the blank** Please complete the proof by justifying why each step is true.

We will solve the integral equation

$$y(x) = 2 + \int_1^x \frac{dt}{ty(t)}, \quad x > 0.$$

We begin by differentiating to obtain $y'(x) = \frac{1}{xy(x)}$.

Note that we know the derivative of the right side by applying the Fundamental Theorem of Calculus.

We now have $\frac{dy}{dx} = \frac{1}{xy}$, which is a separable differential equation.

So we may write, $\int y dy = \int \frac{1}{x} dx$.

From this we obtain $\frac{1}{2}y^2 = \ln(x) + C, \quad x > 0$. The right-side of this equation is the result of knowing the definition of the natural logarithm

Letting $x = 1$ in the original equation, we find that $y(1) =$ 2. Thus C equals 2 and we are able to solve for y. □

3. [9 points each] Determine whether the following series converge or diverge. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

(a) $\sum_{n=1}^{\infty} n^2 3^{-n^2}$

We apply the Ratio Test; note terms are positive.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{3^{(n+1)^2}}}{\frac{n^2}{3^{n^2}}} = \lim_{n\to\infty} \frac{3^{n^2}}{3^{n^2+2n+1}} \cdot \left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{3^{2n+1}}$$

$$= 1 \cdot 0 = 0$$

As $0 < 1$, the Ratio Test implies convergence

(b) $\sum_{n=1}^{\infty} \frac{3-n^2}{(n+3)^3}$

Let's note that this series is equal to $\sum \frac{3}{(n+3)^3} - \sum \frac{n^2}{(n+3)^3}$

The first series is convergent via a comparison to the convergent p-series $\sum \frac{1}{n^3}$.

The 2nd series is divergent via a comparison to the divergent p-series $\sum \frac{1}{n}$.

Thus, the given series is divergent.

(c) $\sum_{n=1}^{\infty} (\frac{n!}{n^n})^2$

Note that $0 \le \frac{n!}{n^n} \le 1$ for all $n \ge 1$. Thus, $0 \le \left(\frac{n!}{n^n}\right)^2 \le \frac{n!}{n^n}$.

As we have seen earlier that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges, by a direct comparison so does the given series.

(Note: to show $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges, one may use the Ratio Test.

See page 760 for a similar example.)

4. [10 points] Determine the interval of convergence for the power series.

$$\sum_{k=0}^{\infty}(-1)^k \frac{(x-5)^k}{3^k(k+1)}$$

We apply the Ratio Test.

$$\lim_{k\to\infty}\left|\frac{(x-5)^{k+1}}{3^{k+1}(k+2)}\cdot\frac{3^k\cdot(k+1)}{(x-5)^k}\right| = \lim_{k\to\infty}\frac{|x-5|}{3}\,\frac{k+1}{k+2} = \frac{|x-5|}{3}\cdot\lim_{k\to\infty}\frac{k+1}{k+2}$$

$$= \frac{|x-5|}{3}.$$

When this ratio is less than 1, we have convergence.

So we compute:

$$\frac{|x-5|}{3} < 1 \iff |x-5| < 3 \iff -3 < x-5 < 3$$

$$\iff 2 < x < 8.$$

We see that the radius of convergence is 3. We test the endpoints of this interval in a different manner.

At $x=2$ the series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k(-3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty}\frac{1}{k+1}$, the divergent harmonic series.

At $x=8$ the series is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k(3)^k}{3^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$, the convergent alternating harmonic series.

Thus, the interval of convergence is $(2, 8]$.

5. Taylor's method

(a) [10 points] Determine the first five terms of the Taylor series of $f(x) = \sqrt{x}$ at $x = 1$.

The Taylor Series at $x=1$ has the following form:

$$c_0 + c_1(x-1) + c_2(x-1)^2 + \ldots$$

We must find c_0, c_1, c_2, c_3, c_4. We compute the derivatives of $f(x) = x^{1/2}$.

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}, \quad f^{4}(x) = \frac{-15}{16}x^{-7/2}$$

So $c_0 = \dfrac{f(1)}{0!} = \dfrac{1^{1/2}}{1} = 1$

$$c_1 = \frac{f'(1)}{1!} = \frac{1/2}{1} = \frac{1}{2}$$

$$c_2 = \frac{f''(1)}{2!} = \frac{-1/4}{2} = -\frac{1}{8}$$

$$c_3 = \frac{f'''(1)}{3!} = \frac{3/8}{3!} = \frac{3}{48}$$

$$c_4 = \frac{f^{4}(1)}{4!} = \frac{-15/16}{4!} = \frac{-15}{384}$$

Thus, the first five terms of the Taylor Series is:

$$1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^3 - \frac{15}{384}(x-1)^4 + \ldots$$

(b) [8 points] Use the third-degree Taylor polynomial to *show how* to approximate $\sqrt{1.5}$ as a decimal. (You needn't do the arithmetic.)

The 3rd-degree Taylor polynomial is

$$T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^3$$

We use this to approximate $\sqrt{1.5}$

$$T_3(1.5) = 1 + \frac{1}{2}\left(+\frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{2}\right)^2 + \frac{3}{48}\left(\frac{1}{2}\right)^3$$

$$= 1 + \frac{1}{4} - \frac{1}{32} + \frac{1}{128}$$

$$= \frac{128 + 32 - 4 + 1}{128} = \frac{157}{128}$$

(c) [7 points] Use Taylor's Inequality to bound the error for the estimate given in the previous step.

Recall the statement of Taylor's Inequality:

If $|f^{n+1}(x)| \le M$ for $|x-a| \le d$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$.

In the previous step we considered $n=3$. We have $a=1$ and require $d \ge 1/2$ since we have $x = 1.5$.

Note $|f^4(x)| = \left| \frac{-15}{16} \cdot \frac{1}{x^{7/2}} \right| = \frac{15}{16 x^{7/2}}$.

On $[1/2, 1 1/2]$ this is largest at $x = 1/2$. So $M = \frac{15}{16(1/2)^{7/2}}$.

Thus, $|R_3(x)| \le \frac{M}{4!}|1.5-1|^4 = \frac{M}{4!\,2^4}$, where M is as above.

That is, our error is bounded by $\frac{M}{4!\,2^4}$.

6. [8 points each] Evaluate each of the following integrals. State the domain of your answer.

(a) $\int \frac{x^2+6x+9}{x^2+6x}dx$

$$\int \frac{x^2+6x+9}{x^2+6x}\,dx = \int 1 + \frac{9}{x^2+6x}\,dx = \int 1 + \frac{9}{x(x+6)}\,dx$$

We perform a partial fraction decomposition on $\frac{9}{x(x+6)}$.

$$\frac{9}{x(x+6)} = \frac{A}{x} + \frac{B}{x+6} \iff 9 = A(x+6) + Bx \Rightarrow \begin{matrix} 0 = A+B \\ 9 = 6A \end{matrix} \Rightarrow A = \frac{3}{2},\ B = -3/2$$

Thus, we have $\int 1 + \frac{3/2}{x} + \frac{-3/2}{x+6}\,dx \quad , x \neq 0,\ x \neq -6$

$$= x + \frac{3}{2}\ln|x| - \frac{3}{2}\ln|x+6| + C = x + \frac{3}{2}\ln\left|\frac{x}{x+6}\right| + C.$$

(b) $\int[\sec(x) - \tan(x)]^2dx$

$$= \int \sec^2 x - 2\sec x\tan x + \tan^2 x\,dx$$

$$= \tan x - 2\sec x + \int \tan^2 x\,dx$$

$$= \tan x - 2\sec x + \int \sec^2 x - 1\,dx$$

$$= 2\tan x - 2\sec x - x + C, \quad x \text{ any angle except } x \neq (2k+1)\pi$$

(c) $\int \frac{1}{\sqrt{9-4x^2}}dx$

We will make a trigonometric substitution.

Let $x = \frac{3}{2}\sin\theta$, then $dx = \frac{3}{2}\cos\theta \, d\theta$ and $\sqrt{9-4x^2} = \sqrt{9-4\cdot\frac{9}{4}\sin^2\theta}$

$$= \sqrt{9(1-\sin^2\theta)}$$

$$= 3\cos\theta.$$

Thus, the integral becomes

$$\int \frac{3/2 \cos\theta}{3\cos\theta} d\theta = \int \frac{1}{2} d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right) + C$$

needs $-1 < \frac{2}{3}x < 1$

(d) $\int \ln(2t)\frac{1}{t}dt$

For $t > 0$, let $u = \ln(2t)$. Then $du = \frac{1}{2t}\cdot 2\,dt = \frac{dt}{t}$.

Thus, $\int \frac{\ln(2t)}{t}dt = \int u\,du = \frac{u^2}{2} + C$

$$= \frac{(\ln(2t))^2}{2} + C.$$

7. [10 points]Carbon dating is a method used by scientists to accurately determine the age of organic matter that may be up to around 50,000 years. When a plant or animal is living, the ratio of radioactive Carbon-14 to ordinary carbon stays constant. After the organism dies, no new carbon is ingested, and the proportion of Carbon-14 in the organisms remains decreases as the Carbon-14 decays. The rate of decay of Carbon-14 is proportional to the amount still present. The half-life of Carbon-14 is 5,700 years.

(a) Define some useful notation and give a differential equation that models the decay of Carbon-14. What initial condition would you hope to have to help make your model more specific?

Let t denote time in years. Let $C(t)=C$ denote the proportion of carbon-14 present at time t.

As the rate of decay is proportional to the amount still present, we have: $\frac{dC}{dt}=kC$, where k is a constant

We would hope to have the knowledge of the proportion of C-14 present in a given organism while it is alive.

(b) Solve the differential equation to find the amount of radioactive Carbon-14 nuclei as a function to time. You may express your answer in terms of the initial amount of radioactive material present when the organism was still alive. Indicate how the stated half-life relates to your model and helps to make it more specific and thus more useful.

$\frac{dC}{dt}=kC \Rightarrow$ by separating variables, $\int\frac{dC}{C}=\int k\,dt$, then by integrating $\ln|C|=kt+A$. We solve for C; $|C|=e^{kt+A} \Rightarrow$

$C=A_0e^{kt}$

We can solve for k knowing the half-life: $\frac{1}{2}=1e^{k\cdot 5700} \Rightarrow \ln\left(\frac{1}{2}\right)=5700k$

$\Rightarrow k=\frac{\ln(1/2)}{5700}$

(c) [Application] An ancient scroll was found by archeologists in the Middle East in a sealed earthen vase, and the well-preserved manuscript contained 70% of the Carbon-14 found in living matter. About how old is the manuscript?

$$.7C_0=C_0e^{\frac{\ln(1/2)}{5700}t}$$

$$.7=e^{\frac{\ln(1/2)}{5700}t}$$

$$\frac{5700\cdot\ln(.7)}{\ln(1/2)}=t$$

$t\approx 2933$ years (via a calculator).

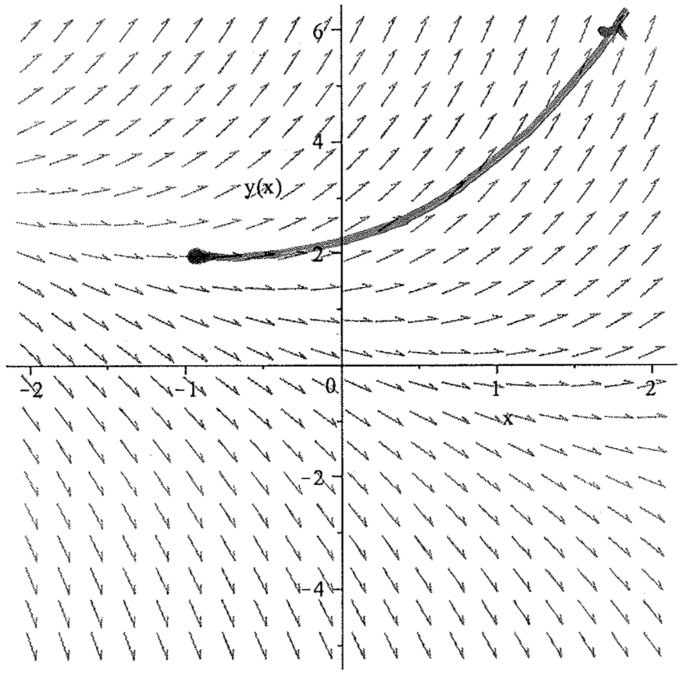

Figure 1: The direction field

8. [10 points] A direction field for the differential equation $y' = x + y - 1$ is shown.

(a) Sketch the graphs of the solutions that satisfy the given initial condition $(-1, 2)$.

See the figure.

(b) Use Euler's method with step size of $h = 0.2$ to estimate $y(-0.6)$.

$y(-1) = 2$

$y(-0.8) = 2 + (.2)(-1 + 2 - 1) = 2 + 0 = 2$

$y(-0.6) = 2 + (.2)(-.8 + 2 - 1) = 2 + .2(.2) = 2.04.$