Name Solution Key
ID number 3.14159 26...
Sections **B**

Calculus II (Math 122) Final Exam, 17 May 2013

This is a closed book exam. No notes or calculators are allowed. A table of trigonometric identities is attached. To receive credit you must show your work. Please leave answers as square roots, ln(), exp(), fractions, or in terms of constants like e, π, etc. Please turn off all cell-phones and other electronic devices. When you are finished please write and sign the honor code (I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.) in the space provided below. Good luck!

1	
2	
3	
4	
5	
6	
7	
8	
Total	85

Honor Code: I have neither given nor received unauthorized aid on this exam. I have not witnessed another giving or receiving unauthorized aid.

Signature:

1. [5 points] Solve the following first-order linear differential equation.

$$y' + y = \sin(e^x)$$

We identify this having the form

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

So we multiply by an integrating factor, $e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$,

to obtain

$$e^x y' + e^x y = e^x \sin(e^x)$$

We recognize the left side as the derivative of a product:

$$\frac{d(e^x y)}{dx} = e^x \sin(e^x)$$

We integrate both sides with respect to x,

$$\int \frac{d(e^x y)}{dx}\,dx = \int e^x \sin(e^x)\,dx$$

We obtain, by an application of FTC on left and a u-sub. on right,

$$e^x y = -\cos(e^x) + C$$

$$\Rightarrow \quad y = \frac{-\cos(e^x) + C}{e^x}$$

2. [1 point each] **Fill in the blank** Please complete the proof by justifying why each step is true.

Claim: If population growth follows the logistic model, then population growth begins to slow when the population reaches half of the carrying capacity.

PROOF: Let us consider the logistic differential equation $\frac{dP}{dt} = rP(M-P)$ for some positive constants r and M.

As we wish to determine when the rate of growth begins to slow, we consider the second derivative. Thus, we will find $\frac{d^2P}{dt^2}$.

So, we compute,

$\frac{d^2P}{dt^2} = rP(-\frac{dP}{dt}) + (M-P)r\frac{dP}{dt}$ by the Product Rule for differentiation.

By a substitution, this equals $rP(-rP(M-P)) + (M-P)r^2P(M-P)$. We may distribute to obtain

$-r^2P^2(M-P) + r^2P(M-P)^2 = -r^2P^2M + r^2P^3 + r^2PM^2 - 2r^2MP^2 + r^2P^3$

collect like terms to obtain

$r^2P\left(2P^2 + M^2 - 3PM\right)$,

and factor, to obtain

$r^2P(2P-M)(P-M)$.

This quantity is zero when $P = 0$, $P = M$ or $P = \frac{M}{2}$. We now restrict P to the open interval $(0, M)$. So we see that concavity changes when $P = \frac{M}{2}$. That is, the first derivative changes from increasing to decreasing at this population level. □

3. [5 points each] Determine whether the following series converge or diverge. You must state or clearly demonstrate what test you are using to determine convergence and justify its use. Heuristic or intuitive reasoning will not get full credit, though it may help you get started.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+5}$

We use the alternating series test. So, we show the ~~terms~~ ① decreasing and ② go to zero.

① Let $f(x)=\frac{\sqrt{x}}{x+5}$, then $f'(x)=\frac{(x+5)\frac{1}{2}x^{-1/2}-\sqrt{x}}{(x+5)^2} = \frac{\frac{1}{2}(x+5)-x}{\sqrt{x}\,(x+5)^2}$

By examining the numerator (note the denominator is positive for all $x>0$), we see that the first derivative is negative when $-\frac{1}{2}x+5<0$, i.e. $10<x$.

② $\lim_{n\to\infty}\frac{\sqrt{n}}{n+5} = \lim_{n\to\infty}\frac{1}{n^{1/2}+5/n^{1/2}} = 0.$

Thus, the series converges.

(b) $\sum_{n=1}^{\infty}\frac{1}{2+\sin n}$

We use the n^{th} term test for divergence.

Note that $-1\leq \sin n \leq 1$ for all n. Thus,

$1 \leq 2+\sin(n) \leq 3$. So, $\frac{1}{2+\sin(n)} \geq \frac{1}{3}$.

Thus, $\frac{1}{2+\sin(n)} \not\to 0$ as $n\to\infty$.

The series diverges.

4. [15 points] Approximate $f(x) = x \ln x$ by a Taylor polynomial of degree $n = 3$ at the number $a = 1$. Then use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $0.5 \le x \le 1.5$.

We will use the method of Taylor.

$f(x) = x \ln x$ $\qquad f(1) = 0$ $\qquad c_0 = \frac{0}{0!} = 0$

$f'(x) = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$ $\qquad f'(1) = 1$ $\qquad c_1 = \frac{1}{1!} = 1$

$f''(x) = \frac{1}{x}$ $\qquad f''(1) = 1$ $\qquad c_2 = \frac{1}{2!} = \frac{1}{2}$

$f'''(x) = \frac{-1}{x^2}$ $\qquad f'''(x) = \frac{-1}{2}$ $\qquad c_3 = \frac{-1/2}{3!} = -1/6$

Thus, $T_3(x) = 0 + 1(x-1)^1 + \frac{1}{2}(x-1)^2 + \frac{-1}{6}(x-1)^3$

To apply Taylor's Inequality, we must find an M that bounds $|f^4(x)|$ on $[0.5, 1.5]$. Note that $f^4(x) = \frac{2}{x^3}$. On the interval $f^4(x)$ is largest when x is smallest, i.e. at $x = 0.5$. Thus $M = \frac{2}{(1/2)^3} = 16$.

So $|R_3(x)| \le \frac{M\,|x-1|^4}{4!}$ on $[0.5, 1.5]$.

$= \frac{16}{24}|x-1|^4$

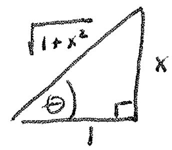
Note that $|x-1|^4$ is largest at $x = 1.5$ (or $x = 0.5$). Thus

$|R_3(x)| \le \frac{2}{3}\left(1/2\right)^4 = \frac{2}{48} = \frac{1}{24}.$

So $T_3(x)$ is w/ $1/24$ of $f(x)$ for all $x \in [0.5, 1.5]$.

5. [5 points each] Evaluate each of the following integrals. State the domain of your answer.

(a) $\int \sqrt{1+x^2}dx$ We do a trigonometric substitution.



Let $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$. So the integrand becomes
where $-\frac{\pi}{2} < \theta < \pi/2$

$$\int \sqrt{1+\tan^2\theta} \cdot \sec^2\theta \, d\theta = \int \sec^3\theta \, d\theta$$

And now we integrate by parts with $u = \sec\theta$ $\quad dv = \sec^2\theta \, d\theta$
$du = \sec\theta\tan\theta \quad v = \tan\theta$

Thus, we obtain

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec\theta \tan^2\theta \, d\theta$$
$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) \, d\theta \text{ by a trig. substitution}$$
$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$
$$\Rightarrow 2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \int \sec\theta \, d\theta$$
$$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$
$$\Rightarrow \int \sec^3\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C\right)$$

We now express in terms of x

$$\frac{1}{2}\left(\sqrt{1+x^2} \cdot x + \ln\left|\sqrt{1+x^2} + x\right| + C\right)$$

(b) $\int x \arctan(x)dx$

We use integration by parts.

Let $u = \arctan x \quad dv = x\,dx$

$du = \frac{1}{1+x^2}dx \quad v = \frac{x^2}{2}$

Then

$$\int x \arctan x \, dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

To integrate $\int \frac{x^2}{1+x^2}dx$, we first divide x^2 by $1+x^2$ to obtain $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$

Thus we have

$$\int x \arctan x \, dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\left[\int 1\,dx - \int \frac{dx}{x^2+1}\right]$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C.$$

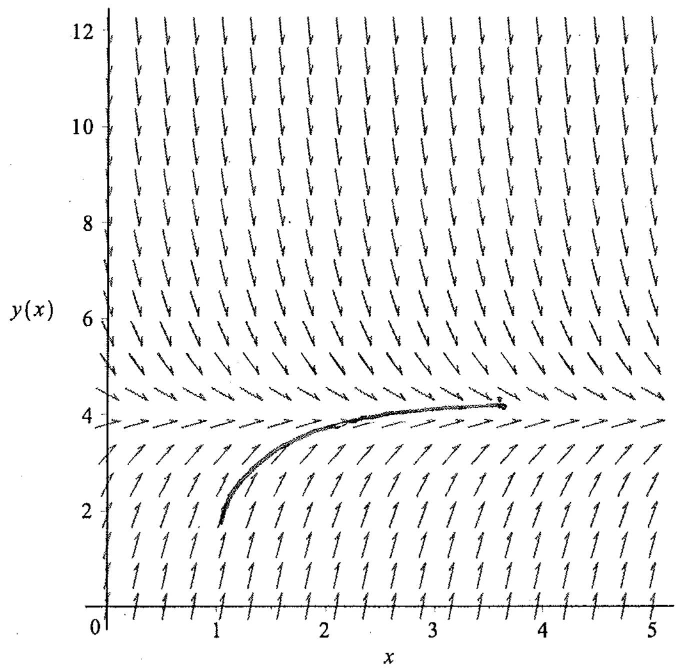


Figure 1: The direction field

6. [15 points] A portion of the direction field for the differential equation $\frac{dy}{dx} = 12 - 3y$ is shown.

   (a) Complete the sketch of the direction field for the portion that has been "covered over". That is, sketch the direction field for $0 \leq x \leq 5$ and $4 \leq y \leq 12$. Shown above

   (b) Sketch the graph of the solution that satisfies the given initial condition $(1, 2)$. Show above.

   (c) Use Euler's method with step size of $h = 0.2$ to estimate $y(1.6)$ (using the same initial conditions as above).

We have $x_0 = 1$, $y_0 = 2$. We must find y_3.

$y_1 = y_0 + h \cdot F(x_0, y_0) = 2 + (0.2)(6) = 3.2$

$\Rightarrow (x_1, y_1) = (1.2, 3.2)$

$y_2 = y_1 + hF(x_1, y_1) = 3.2 + (0.2)(2.4) = 3.68$

$\Rightarrow (x_2, y_2) = (1.4, 3.68)$

$y_3 = y_2 + hF(x_2, y_2) = 3.68 + (0.2)(12 - 3(3.68)) = 3.872$

$\Rightarrow y(1.6) = 3.872$.

7. [10 points] The differential equation in the previous problem is a separable differential equation, so it can be solved analytically. Solve the differential equation subject to the initial condition given in the previous problem.

As the differential equation is separable, we may "separate variables" and write,

$$\int \frac{dy}{12-3y} = \int dx.$$

We solve for y in terms of x.

$$-\frac{1}{3}\int \frac{-3\,dy}{12-3y} = \int dx$$

integrating

$$-\frac{1}{3}\ln|12-3y| = x + C$$

using the exponential function

$$\ln|12-3y| = -3x + \hat{C}$$

$$|12-3y| = e^{-3x+\hat{C}}$$

as both sides are positive

$$12-3y = C_0 e^{-3x}$$

$$12 - C_0 e^{-3x} = 3y$$

$$\frac{12 - C_0 e^{-3x}}{3} = y$$

Knowing that when $x=1, y=2$, we may solve for C_0.

$$\frac{12 - C_0 e^{-3}}{3} = 2 \Rightarrow 12 - C_0 e^{-3} = 6 \Rightarrow 6 = C_0 e^{-3}$$

$$\Rightarrow C_0 = 6e^{3}$$

Thus,

$$y = \frac{12 - 6e^{3}e^{-3x}}{3} = \frac{12 - 6e^{-3x+3}}{3} = 4 - 2e^{-3x+3}$$

8. [10 points] Now solve the same differential equation by finding the Taylor series for y in powers of $(x-1)$, with initial condition $(1,2)$.

The Taylor series for y in powers of $(x-1)$ has the form

$$y = y(1) + \frac{y'(1)}{1!}(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \dots$$

So we must find $y(1), y'(1), \dots$

We have that $y(1) = 2$; this is the given initial condition.

As $y' = 12 - 3y$, $y'(1) = 12 - 3y(1) = 12 - 3\cdot 2 = 6 = 3\cdot 2$

To find $y''(1)$, we first find $y'' = -3\frac{dy}{dx} = -3y'$. So $y''(1) = -3y'(1)$
$= -3\cdot 6 = -18.$
$= -3^2\cdot 2$

To find $y'''(1)$, $y''' = -3y''$. So $y'''(1) = -3(-18) = 54$
$= 3^3 \cdot 2$

and so on, $y^n(1) = (-1)^{n+1} 3^n \cdot 2$.

So we have

$$y = 2 + \frac{6}{1!}(x-1)^1 - \frac{18}{2!}(x-1)^2 + \frac{54}{3!}(x-1)^3 - \dots$$

$$= 2 + 2\sum_{n=1}^{\infty} \frac{3^n (x-1)^n (-1)^{n+1}}{n!}$$

$$= 2 + 2 + 2\sum_{n=0}^{\infty} \frac{(3x-3)^n (-1)^{n+1}}{n!}$$

$$= 4 - 2\sum_{n=0}^{\infty} \frac{(3x-3)^n (-1)^n}{n!} = 4 - 2\sum_{n=0}^{\infty} \frac{(3-3x)^n}{n!}$$

$$= 4 - 2e^{-3x+3}$$