Linear Algebra
Exam 2 - Fall 2021

November 11, 2021

Total
60 points

Average
$\frac{52}{60}$

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [5 points] An **isomorphism** between two vector spaces V and W is a mapping that has three properties. List the three properties of this mapping.

An isomorphism between vector spaces V and W is a

(1) one-to-one

(2) linear transformation

(3) from V onto W.

2. [8 points] Use the fact that $det(AB) = det(A)det(B)$ to compute $det(C^5)$ where

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

By the given fact, we have $\det(C^5) = (\det C)^5$.

So, we must compute $\det C$; there are many ways to do this. One way that we employ is co-factor expansion across the first row:

$$\det C = 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 1(1-4) - 0 + 1(2-1)$$

$$= -3 - 0 + 1 = -2.$$

Thus,

$$\det(C^5) = -2^5 = -32.$$

4. [5 points] Show that the following set is not a subspace of $\mathbb{R}^2$.

$$W = \{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0\}$$

We will show that W is not closed under vector addition.

(It is closed under scalar multiplication and it does contain the zero vector.)

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ since $1 \cdot 0 = 0$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$ since $0 \cdot 1 = 0$.

However, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W$ since $1 \cdot 1 \neq 0$.

5. [5 points] Is the following vector **b** in the null space of C? Let

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

We check to see if $C\vec{b} = \vec{0}$.

$$C\vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \neq \vec{0}.$$

So, NO, $\vec{b} \notin \text{Nul } A$.

OR

In a previous problem we showed $\det C \neq 0$. Therefore C is invertible and by IMT $\dim \text{Nul } C = 0$. Thus, $\vec{b} \notin \text{Nul } C$.

3. [9 points] Suppose we are given the following equation involving block-partitioned matrices

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Suppose that the partition of the matrices is conformable for block multiplication, that all sub-matrices are square and where I stands for the identity matrix.

Find formulas for the following matrices (in the suggested order) in terms of A, B and C:

(a) Find a formula for X.

(b) Find a formula for Z.

(c) FInd a formula for Y.

Let's compute the product on the left and equate.

We get

① $XA + 0B = I$ or $XA = I$

② $X0 + 0C = 0$ or $0 = 0$

③ $YA + ZB = 0$

④ $Y0 + ZC = I$ or $ZC = I$.

So, by ①, we have $X = A^{-1}$

by ④, we have $Z = C^{-1}$

so by ③ we have $YA = -ZB$

$\Rightarrow YA = -C^{-1}B$

$Y = -C^{-1}BA^{-1}$

Order matters! That is, many people wrote equation 3, for example, incorrectly as $AY + BZ = 0$. Then they ~~incorrectly~~ found what Y is and got the wrong answer.

6. [2 points each] **Fill-in-the-blank**

(a) For an $m \times n$ matrix A, $ColA = \mathbb{R}^m$ if an only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution

for every $\vec{b} \in \mathbb{R}^m$.

(b) The dimension of $NulA$ is the

number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

(c) In a p-dimensional vector space V, any linearly independent set of p vectors is automatically

a spanning set and thus a basis for V.

(d) For bases $\mathcal{B}$ and $\mathcal{C}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ can be found by performing Gaussian elimination on the following augmented matrix

$[\vec{c}_1 \dots \vec{c}_n \mid \vec{b}_1 \dots \vec{b}_n]$

until arriving at

$[\, I_n \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \,]$

7. [5 points] Suppose that H is a 1-dimensional subspace of the vector space $\mathbb{R}^3$. Consider a set of 3 vectors $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ that spans H. The Spanning Set Theorem states that if one of the vectors in S is a linear combination of the remaining vectors in S, then the set formed from S by removing it still spans H. Suppose that $\mathbf{b_2} = 2\mathbf{b_1}$ and that

$$\mathbf{b_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

If $Span\{\mathbf{b_3}\}$ is not H, what can you say about $\mathbf{b_3}$? Why does this not contradict the Spanning Set Theorem?

This is a little tricky!

As H is 1-dimensional, $\vec{b_3} \in H$ but $Span\{\vec{b_3}\} \neq H$, we can say $\vec{b_3} = \vec{0}$.

The Spanning Set Theorem would say that some subset of size 1 contained in the set is a basis, but not every subset of size 1 needs to be.

8. [8 points] Let $\mathbf{p_1} = 1 + t$, $\mathbf{p_2} = t + t^2$, and $\mathbf{p_3} = 1 + t + t^2$. Use coordinate vectors to show that these polynomials form a basis $\mathcal{B}$ for $\mathbb{P}_2$.

By a coordinate mapping, these vectors in $\mathbb{R}^3$ are

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We use Gaussian Elimination on $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ to show that this matrix is row equivalent to I_3. So, by the Invertible Matrix Theorem these column vectors are a linearly independent spanning set, i.e. a basis. Thus the original ones are too.

Note: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find $\mathbf{q}$ in $\mathbb{P}_2$, given that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

To do this, we solve: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

Thus, $q = 2 + 3t + 2t^2$.

9. [7 points] Suppose that a 6×8 matrix has four pivot columns. It is technically wrong to say that $Col A$ is *equal* to $\mathbf{R}^4$. What can you say? And, what vector space contains $Col A$?

We can say that Col A is isomorphic to $\mathbb{R}^4$ and is a subspace of $\mathbb{R}^6$.

How many vectors are in a basis for $Nul A$? Why?

As A has 8 columns, four of which are pivot columns, there are 4 columns which are not pivot columns. By the Rank Theorem dim Nul A is 4, that is a basis for Nul A has 4 vectors.